Large graph limit for an SIR process in random network with heterogeneous connectivity

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Abstract

We consider a SIR epidemic model propagating on a random network generated by configuration model, where the degree distribution of the vertices is given and where the edges are randomly matched. The evolution of the epidemics is summed up into three measure-valued equations that describe the degrees of the susceptible individuals and the number of edges from an infectious or removed individual to the set of susceptibles. These three degree distributions are sufficient to describe the course of the disease. The limit in large population is investigated. As a corollary, this provides a rigorous proof of the equations obtained by Volz [27].

Keywords: Configuration model graph, SIR model, measure-valued process, large network limit.
AMS codes: 60J80, 05C80, 92D30, 60F99.

1 Introduction and notation

In this work, we investigate an epidemic spreading on a random graph with fixed degree distribution and evolving according to an SIR model as follows. Every individual not yet infected is assumed to be susceptible. Infected individuals stay infected during random exponential times with mean $1/\beta$ during which they infect each of their susceptible neighbors with rate r. At the end of the infectious period, the individual becomes removed and is no longer susceptible to the disease. Contrarily to the classical mixing compartmental SIR epidemic models (*e.g.* [17, 5] see also [2] Chapter 2 for a presentation), heterogeneity in the number of contacts makes it difficult to describe the dynamical behavior of the epidemic. Mean field approximations (*e.g.* [23, 4, 10]) or large population approximations (*e.g.* [3], see also Eq. (3) of [1] in discrete time) provide a set of denumerable equations to describe our system. We are here inspired by the paper of Volz [27], who proposes a low-dimensional system of five differential equations for the dynamics of an SIR model on a Configuration Model (CM) graph [7, 19]. We refer to Volz' article for a bibliography about SIR models on graphs (see also Newman [20, 21], Durrett [10] or Barthélemy *et al.* [4]). Starting from a random model in finite population, Volz derives deterministic equations by increasing the size of the network, following in this respect works of Newman for instance ([21]). The convergence of the continuous-time stochastic SIR model to its deterministic limit for large graphs was however not proved. In this paper, we prove the convergence that was left open by Volz. To achieve this, we provide a rigorous individual-based description of the epidemic on a random graph. Three degree distributions are sufficient to describe the epidemic dynamics. We describe these distributions by equations in the space of measures on the set of nonnegative integers, of which Volz' equations are a by-product. Starting with a node-centered description, we show that the individual dimension is lost in the large graph limit. Our construction heavily relies on the choice of a CM for the graph underlying the epidemic, which was also made in [27].

The size N of the population is fixed. The individuals are related through a random network and are represented by the *vertices* of an undirected graph. Between two neighbors, we place an *edge*. The graph is non-oriented and an edge between x and y can be seen as two directed edges, one from x to y and the other from y to x. If we consider an edge as emanating from the vertex x and directed to the vertex y, we call x the *ego* of the edge and y the *alter*. The number of neighbors of a given individual is the *degree* of the associated vertex. The degree of x is denoted d_x. It varies from an individual to another one. The CM developed in Section 2.1 is a random graph where individuals' degrees are independent random variables with same distribution $(p_k)_{k\in\mathbb{N}}$. Edges are paired at random. As a consequence, for a given edge, alter has the size-biased degree distribution: the probability that her degree is k is $kp_k/\sum_{\ell\in\mathbb{N}}\ell p_\ell$.
The population is partitioned into the classes of susceptible, infectious or removed individuals. At time t, we denote by S_t, I_t and R_t the set of susceptible, infectious and removed nodes. We denote by S_t, I_t and R_t the sizes of these classes at time t. With a slight abuse, we will say that a susceptible individual is of type S (accordingly of type I or R) and that an edge linking an infectious ego and susceptible alter is of type IS (accordingly RS, II or IR). For $x \in \mathrm{I}$ (respectively R), $d_x(\mathrm{S})$ represents the number of edges with x as ego and susceptible alter. The numbers of edges with susceptible ego (resp. of edges of types IS and RS) are denoted by N_t^{S} (resp. N_t^{IS} and N_t^{RS}).
A possible way to describe rigorously the epidemics' evolution is given in Section 2.2. We consider the subgraph of infectious and removed individuals with their degrees. Upon infection, the infectious ego chooses the edge of a susceptible alter at random. Hence the latter individual is chosen proportionally to her degree. When she is connected, she uncovers the edges to neighbors that were already in the subgraph.

We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. The space of real bounded functions on $\mathbb{N}$ is denoted by $\mathcal{B}_b(\mathbb{N})$. For any $f\in\mathcal{B}_b(\mathbb{N})$, set $\|f\|_\infty$ the supremum of f on $\mathbb{N}$. For all such f and $y\in\mathbb{N}$, we denote by $\tau_y f$ the function $x\mapsto f(x-y)$. For all $n\in\mathbb{N}$, χ^n is the function $x\mapsto x^n$, and in particular, $\chi\equiv\chi^1$ is the identity function, and $\mathbf{1}\equiv\chi^0$ is the function constantly equal to 1.

We denote by $\mathcal{M}_F(\mathbb{N})$ the set of finite measures on $\mathbb{N}$, embedded with the topology of weak convergence. For all $\mu\in\mathcal{M}_F(\mathbb{N})$ and $f\in\mathcal{B}_b(\mathbb{N})$, we write

$$\langle \mu, f\rangle = \sum_{k\in\mathbb{N}} f(k)\,\mu(\{k\}).$$

With some abuse of notation, for all $\mu\in\mathcal{M}_F(\mathbb{N})$ and $k\in\mathbb{N}$, we denote $\mu(k)=\mu(\{k\})$. For $x\in\mathbb{N}$, we write δ_x for the Dirac measure at point x. Note, that some additional notation is provided in Appendix A, together with several topological results, that will be used in the sequel.

The plan of the paper and the main results are described below. In Section 2, we describe the mechanisms underlying the propagation of the epidemic on the CM graph. To describe the course of the epidemic, rather than the sizes S_t, I_t and R_t, we consider three degree distributions given as point measures of $\mathcal{M}_F(\mathbb{N})$, for $t\geq 0$:

$$\mu_t^{\mathrm{S}} = \sum_{x\in\mathrm{S}_t}\delta_{d_x},\quad \mu_t^{\mathrm{IS}} = \sum_{x\in\mathrm{I}_t}\delta_{d_x(\mathrm{S}_t)},\quad \mu_t^{\mathrm{RS}} = \sum_{x\in\mathrm{R}_t}\delta_{d_x(\mathrm{S}_t)}. \tag{1.1}$$

Notice that the measures μ_t^{S}/S_t, μ_t^{IS}/I_t and μ_t^{RS}/R_t are probability measures that correspond to the usual (probability) degree distribution. The degree distribution μ_t^{S} of susceptible individuals is needed to describe the degrees of the new infected individuals. The measure μ_t^{IS}

provides information on the number of edges from I_t to S_t, through which the disease can propagate. Similarly, the measure μ_t^{RS} is used to describe the evolution of the set of edges linking S_t to R_t. We can see that $N_t^{\mathrm{S}} = \langle \mu_t^{\mathrm{S}}, \chi\rangle$ and $S_t = \langle \mu_t^{\mathrm{S}}, \mathbf{1}\rangle$ (and accordingly for N_t^{IS}, N^{RS}, I_t and R_t).

In Section 3, we study the large graph limit obtained when the number of vertices tends to infinity, the degree distribution being unchanged. The degree distributions mentioned above can then be approximated, after proper scaling, by the solution $(\bar{\mu}_t^{\mathrm{S}}, \bar{\mu}_t^{\mathrm{IS}}, \bar{\mu}_t^{\mathrm{RS}})_{t\geq 0}$ of the system of deterministic measure-valued Equations (1.3)-(1.5) with initial conditions $\bar{\mu}_0^{\mathrm{S}}$, $\bar{\mu}_0^{\mathrm{IS}}$ and $\bar{\mu}_0^{\mathrm{RS}}$.
For all $t\geq 0$, we denote by $\bar{N}_t^{\mathrm{S}} = \langle \bar{\mu}_t^{\mathrm{S}}, \chi\rangle$ (resp. $\bar{N}_t^{\mathrm{IS}} = \langle \bar{\mu}_t^{\mathrm{IS}}, \chi\rangle$ and $\bar{N}_t^{\mathrm{RS}} = \langle \bar{\mu}_t^{\mathrm{RS}}, \chi\rangle$) the continuous number of edges with ego in S (resp. IS edges, RS edges). Following Volz [27], pertinent quantities are the proportions $\bar{p}_t^{\mathrm{I}} = \bar{N}_t^{\mathrm{IS}}/\bar{N}_t^{\mathrm{S}}$ (resp. $\bar{p}_t^{\mathrm{R}} = \bar{N}_t^{\mathrm{RS}}/\bar{N}_t^{\mathrm{S}}$ and $\bar{p}_t^{\mathrm{S}} = (\bar{N}_t^{\mathrm{S}} - \bar{N}_t^{\mathrm{IS}} - \bar{N}_t^{\mathrm{RS}})/\bar{N}_t^{\mathrm{S}}$) of edges with infectious (respectively removed, susceptible) alter among those having susceptible ego.
We also introduce

$$\theta_t = \exp\Big(-r\int_0^t \bar{p}_s^{\mathrm{I}}\,\mathrm{d}s\Big) \tag{1.2}$$

the probability that a degree one node remains susceptible until time t. For any $f\in\mathcal{B}_b(\mathbb{N})$,

$$\langle \bar{\mu}_t^{\mathrm{S}}, f\rangle = \sum_{k\in\mathbb{N}} \bar{\mu}_0^{\mathrm{S}}(k)\,\theta_t^k f(k), \tag{1.3}$$

$$\begin{aligned}\langle \bar{\mu}_t^{\mathrm{IS}}, f\rangle = {} & \langle \bar{\mu}_0^{\mathrm{IS}}, f\rangle - \int_0^t \beta\langle \bar{\mu}_s^{\mathrm{IS}}, f\rangle\,\mathrm{d}s \\ & + \int_0^t \sum_{k\in\mathbb{N}} rk\bar{p}_s^{\mathrm{I}} \sum_{\substack{j,\ell,m\in\mathbb{N}\\ j+\ell+m=k-1}} \binom{k-1}{j,\ell,m} (\bar{p}_s^{\mathrm{I}})^j(\bar{p}_s^{\mathrm{R}})^\ell(\bar{p}_s^{\mathrm{S}})^m f(m)\bar{\mu}_s^{\mathrm{S}}(k)\,\mathrm{d}s \\ & + \int_0^t \sum_{k\in\mathbb{N}} rk\bar{p}_s^{\mathrm{I}}(1+(k-1)\bar{p}_s^{\mathrm{I}}) \sum_{k'\in\mathbb{N}^*} \big(f(k'-1)-f(k')\big)\frac{k'\bar{\mu}_s^{\mathrm{IS}}(k')}{\bar{N}_s^{\mathrm{IS}}}\bar{\mu}_s^{\mathrm{S}}(k)\,\mathrm{d}s,\end{aligned} \tag{1.4}$$

$$\begin{aligned}\langle \bar{\mu}_t^{\mathrm{RS}}, f\rangle = {} & \langle \bar{\mu}_0^{\mathrm{RS}}, f\rangle + \int_0^t \beta\langle \bar{\mu}_s^{\mathrm{IS}}, f\rangle\,\mathrm{d}s \\ & + \int_0^t \sum_{k\in\mathbb{N}} rk\bar{p}_s^{\mathrm{I}}(k-1)\bar{p}_s^{\mathrm{R}} \sum_{k'\in\mathbb{N}^*} \big(f(k'-1)-f(k')\big)\frac{k'\bar{\mu}_s^{\mathrm{RS}}(k')}{\bar{N}_s^{\mathrm{RS}}}\bar{\mu}_s^{\mathrm{S}}(k)\,\mathrm{d}s.\end{aligned} \tag{1.5}$$

We denote by $\bar{S}_t$ (resp. $\bar{I}_t$ and $\bar{R}_t$) the mass of the measure $\bar{\mu}_t^{\mathrm{S}}$ (resp. $\bar{\mu}_t^{\mathrm{IS}}$ and $\bar{\mu}_t^{\mathrm{RS}}$). As for the finite graph, $\bar{\mu}_t^{\mathrm{S}}/\bar{S}_t$ (resp. $\bar{\mu}_t^{\mathrm{IS}}/\bar{I}_t$ and $\bar{\mu}_t^{\mathrm{RS}}/\bar{R}_t$) is the probability degree distribution of the susceptible individuals (resp. the probability distribution of the degrees of the infectious and removed individuals towards the susceptible ones).
Let us give an heuristic explanation of Equations (1.3)-(1.5). Remark that the graph in the limit is infinite. The probability that an individual of degree k has been infected by none of her k edges is θ_t^k and Equation (1.3) follows. In Equation (1.4), the first integral corresponds to infectious individuals being removed. In the second integral, $rk\bar{p}_s^{\mathrm{I}}$ is the rate of infection of a given susceptible individual of degree k. Once she gets infected, the multinomial term determines the number of edges connected to susceptible, infectious and removed neighbors. Multi-edges do not occur. Each infectious neighbor has a degree chosen in the size-biased distribution $k'\bar{\mu}^{\mathrm{IS}}(k')/\bar{N}^{\mathrm{IS}}$ and the number of edges to S_t is reduced by 1. This explains the third integral. Similar arguments hold for Equation (1.5).

Choosing $f(k) = \mathbb{1}_i(k)$, we obtain the following countable system of ordinary differential equations (ODEs).

$$\bar{\mu}_t^{\mathrm{S}}(i) = \bar{\mu}_0^{\mathrm{S}}(i)\theta_t^i,$$

$$\begin{aligned}\bar{\mu}_t^{\mathrm{IS}}(i) = \bar{\mu}_0^{\mathrm{IS}}(i) + \int_0^t \Bigg\{ & r\bar{p}_s^{\mathrm{I}} \sum_{j,\ell\geq 0}(i+j+\ell+1)\bar{\mu}_s^{\mathrm{S}}(i+j+\ell+1)\binom{i+j+\ell}{i,j,\ell}(\bar{p}_s^{\mathrm{S}})^i(\bar{p}_s^{\mathrm{I}})^j(\bar{p}_s^{\mathrm{R}})^\ell \\ & + \Big(r(\bar{p}_s^{\mathrm{I}})^2\langle\bar{\mu}_s^{\mathrm{S}},\chi^2-\chi\rangle + r\bar{p}_s^{\mathrm{I}}\langle\bar{\mu}_s^{\mathrm{S}},\chi\rangle\Big)\frac{(i+1)\bar{\mu}_s^{\mathrm{IS}}(i+1) - i\bar{\mu}_s^{\mathrm{IS}}(i)}{\langle\bar{\mu}_s^{\mathrm{IS}},\chi\rangle} - \beta\bar{\mu}_s^{\mathrm{IS}}(i)\Bigg\}\,\mathrm{d}s,\end{aligned}$$

$$\bar{\mu}_t^{\mathrm{RS}}(i) = \bar{\mu}_0^{\mathrm{RS}}(i) + \int_0^t \Bigg\{\beta\bar{\mu}_s^{\mathrm{IS}}(i) + r\bar{p}_s^{\mathrm{I}}\langle\bar{\mu}_s^{\mathrm{S}},\chi^2-\chi\rangle\bar{p}_s^{\mathrm{R}}\frac{(i+1)\bar{\mu}_s^{\mathrm{RS}}(i+1) - i\bar{\mu}_s^{\mathrm{RS}}(i)}{\langle\bar{\mu}_s^{\mathrm{RS}},\chi\rangle}\Bigg\}\,\mathrm{d}s, \tag{1.6}$$

It is noteworthy to say that this system is similar but not identical to that in Ball and Neal [3]. Our equations differ since our mechanism is not the same (compare Section 2.2 with Section 5 in [3]). We emphasize that the number of links of an individual to S decreases as the epidemic progresses, which modifies her infectivity.
The system (1.3)-(1.5) allows us to recover the equations proposed by Volz [27, Table 3, p.297]. More precisely, the dynamics of the epidemic is obtained by solving the following closed system of four ODEs, referred to as Volz' equations in the sequel. The latter are obtained directly from (1.3)-(1.5) and the definitions of $\bar{S}_t$, $\bar{I}_t$, $\bar{p}_t^{\mathrm{I}}$ and $\bar{p}_t^{\mathrm{S}}$ which relate these quantities to the measures $\bar{\mu}_t^{\mathrm{S}}$ and $\bar{\mu}_t^{\mathrm{IS}}$. Let

$$g(z) = \sum_{k\in\mathbb{N}} \bar{\mu}_0^{\mathrm{S}}(k)z^k \tag{1.7}$$

be the generating function for the initial degree distribution of the susceptible individuals $\bar{\mu}_0^{\mathrm{S}}$, and let $\theta_t = \exp(-r\int_0^t \bar{p}_s^{\mathrm{I}}\,\mathrm{d}s)$. Then, the epidemic can be approximated by the solution of the four following ODEs:

$$\bar{S}_t = \langle\bar{\mu}_t^{\mathrm{S}},\mathbf{1}\rangle = g(\theta_t), \tag{1.8}$$

$$\bar{I}_t = \langle\bar{\mu}_t^{\mathrm{IS}},\mathbf{1}\rangle = \bar{I}_0 + \int_0^t \Big(r\bar{p}_s^{\mathrm{I}}\theta_s g'(\theta_s) - \beta\bar{I}_s\Big)\,\mathrm{d}s, \tag{1.9}$$

$$\bar{p}_t^{\mathrm{I}} = \bar{p}_0^{\mathrm{I}} + \int_0^t \Big(r\,\bar{p}_s^{\mathrm{I}}\bar{p}_s^{\mathrm{S}}\theta_s\frac{g''(\theta_s)}{g'(\theta_s)} - r\,\bar{p}_s^{\mathrm{I}}(1-\bar{p}_s^{\mathrm{I}}) - \beta\bar{p}_s^{\mathrm{I}}\Big)\,\mathrm{d}s, \tag{1.10}$$

$$\bar{p}_t^{\mathrm{S}} = \bar{p}_0^{\mathrm{S}} + \int_0^t r\bar{p}_s^{\mathrm{I}}\bar{p}_s^{\mathrm{S}}\Big(1 - \theta_s\frac{g''(\theta_s)}{g'(\theta_s)}\Big)\,\mathrm{d}s. \tag{1.11}$$

Here, the graph structure appears through the generating function g. In (1.9), we see that the classical contamination terms $r\bar{S}_t\bar{I}_t$ (mass action) or $r\bar{S}_t\bar{I}_t/(\bar{S}_t+\bar{I}_t)$ (frequency dependence) of mixing SIR models (*e.g.* [2, 9]) are replaced by $r\bar{p}_t^{\mathrm{I}}\theta_t g'(\theta_t) = r\bar{N}_t^{\mathrm{IS}}$. The fact that new infectious individuals are chosen in the size-biased distribution is hidden in the term $g''(\theta_t)/g'(\theta_t)$.

The beginning of the epidemic and computation of the reproduction number, when the numbers of infected individuals and of contaminating edges are small and when Volz's deterministic approximation does not hold, makes the object of another study.

2 SIR model on a Configuration Model graph

In this section, we introduce Configuration Model graphs and describe the propagation of SIR on such graphs.

2.1 Configuration Model graph

Graphs at large can be mathematically represented as matrices with integer entries: to each graph corresponds an adjacency matrix, the (x, y)-th coefficient of which is the number of edges between the vertices x and y. Defining the distribution of a random graph thus amounts to choosing a sigma-field and a probability measure on the space $\mathbb{N}^{\mathbb{N}^*\times\mathbb{N}^*}$, where $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. Another approach is to construct a random graph by modifying progressively a given graph, as in Erdös-Renyi model. Several other constructions are possible such as the preferential attachment model, the threshold graphs, etc.

Here, we are interested in the Configuration Model (CM) proposed by Bollobás [7], Molloy and Reed [19] (see also [21, 22, 10, 26]) and which models graphs with specified degree distribution and independence between the degrees of neighbors. As shown by statistical tests, these models might be realistic in describing community networks. See for instance Clémençon *et al.* [8] for dealing with the spread of the HIV-AIDS disease among the homosexual community in Cuba.
We recall its construction (see *e.g.* [10, 26]). Suppose we are given the number of vertices, N, and i.i.d. random variables (r.v.) $d_1, \dots, d_N$ with distribution $(p_k)_{k\in\mathbb{N}}$ that represent the degrees of each vertex. To the vertex i are associated d_i half-edges. To construct an edge, one chooses two open half-edges uniformly at random and pair them together.
Remark that this linkage procedure does not exclude self-loops or multiple edges. In the following, we are interested in a large number of nodes with a fixed degree distribution, hence self-loops and multiple edges become less and less apparent in the global picture (see *e.g.* [10, Theorem 3.1.2]).
Notice that the condition for the existence with positive probability of a giant component is that the expectation of the size biased distribution is larger than 1:

$$\sum_{k\in\mathbb{N}}(k-1)\frac{kp_k}{\sum_{\ell\in\mathbb{N}}\ell p_\ell} > 1.$$

This is connected with the fact that the Galton-Watson tree with this offspring distribution is supercritical (see [10, Section 3.2 p. 75] for details).

2.2 SIR epidemic on a CM graph

We now propagate an epidemic on a CM graph of size N (see Figure 1). The disease can be transmitted from infectious nodes to neighboring susceptible nodes and removed nodes cannot be reinfected.

Suppose that at initial time, we are given a set of *susceptible* and *infectious* nodes together with their degrees. The graph of relationships between the individuals is in fact irrelevant for studying the propagation of the disease. The minimal information consists in the sizes of the classes S, I, R and the number of edges to the class S for every infectious or removed node. Thus, each node of class S comes with a given number of half-edges of undetermined types ; each node of class I (resp. R) comes with a number of IS (resp. RS) edges. The numbers of IR, II and RR edges need not to be retained.

The evolution of the SIR epidemic on a CM graph can be described as follows. To each IS-type half-edge is associated an independent exponential clock with parameter r and to each I vertex is associated an independent exponential clock with parameter β. The first of all these clocks that rings determines the next event.

Case 1 If the clock that rings is associated to an I individual, the latter recovers. Change her status from I to R and the type of her emanating half-edges accordingly: IS half-edges become RS half-edges.

Case 2 If the clock is associated with a half IS-edge, an infection occurs.

Step 1 Match randomly the IS-half-edge that has rung to a half-edge belonging to a susceptible.

Step 2 This susceptible is the newly infected. Let k be her degree. Choose uniformly $k-1$ half-edges among all the available half-edges (they either are of type IS, RS, or emanate from S). Let m (resp. j and ℓ) be the number of SS-type (resp. of IS and of RS-type) half-edges drawn among these $k-1$ half-edges;

Step 3 The chosen half-edges of type IS and RS determine the infectious or removed neighbors of the newly infected individual. The remaining m edges of type SS remain open in the sense that the susceptible neighbor is not fixed. Change the status of the m (resp. j, ℓ) SS-type (resp. IS-type, RS-type) edges created to SI-type (resp. II-type, RI-type);

Step 4 Change the status of the newly infected from S to I. $\square$

We then wait for another clock to ring and repeat the procedure.

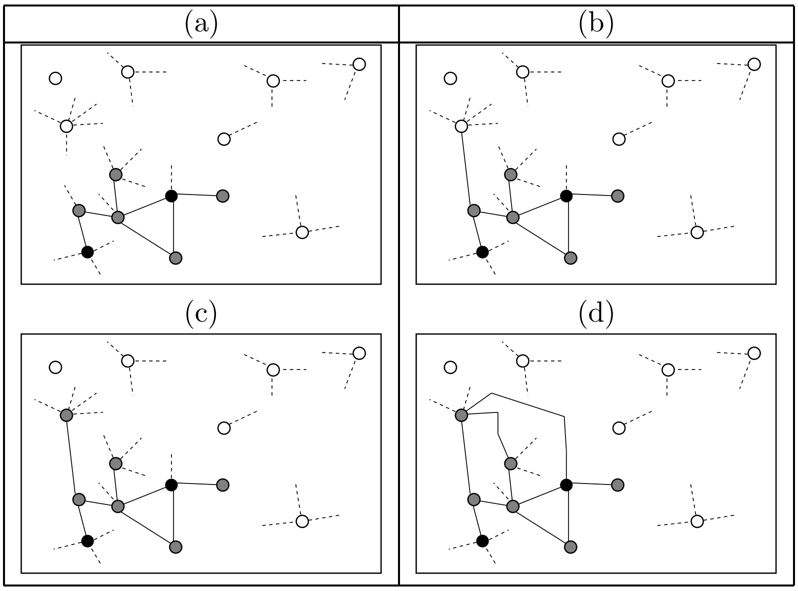

Figure 1: *Infection process. Arrows provide the infection tree. Susceptible, infectious and removed individuals are colored in white, grey and black respectively. (a) The degree of each individual is known, and for each infectious (resp. removed) individual, we know her number of edges of type* IS *(resp.* RS*). (b-c) A contaminating half-edge is chosen, and say that a susceptible of degree k is infected at time t. The contaminating edge is drawn in bold line. (d) Once the susceptible individual has been infected, we determine how many of her remaining arrows are linked to the classes* I *and* R*. If we denote by j and ℓ these numbers, then $N^{\mathrm{IS}}_t = N^{\mathrm{IS}}_{t_-} - 1 + (k-1) - j - \ell$ and $N^{\mathrm{RS}}_t = N^{\mathrm{RS}}_{t_-} - \ell$.*

We only need three descriptors of the system to obtain a Markovian evolution, namely the three degree distributions introduced in (1.1).

For a measure $\mu \in \mathcal{M}_F(\mathbb{N})$, we denote by $F_\mu(m) = \mu(\{0,\dots,m\})$, $m \in \mathbb{N}$, its cumulative distribution function. We introduce F_μ^{-1} its right inverse (see Appendix A). Then, for all $0 \le i \le S_t$ (resp. $0 \le i \le I_t$ and $0 \le i \le R_t$),

$$\gamma_i(\mu_t^{\mathrm{S}}) = F^{-1}_{\mu_t^{\mathrm{S}}}(i), \qquad \Big(\text{resp. } \gamma_i(\mu_t^{\mathrm{IS}}) = F^{-1}_{\mu_t^{\mathrm{IS}}}(i), \qquad \gamma_i(\mu_t^{\mathrm{RS}}) = F^{-1}_{\mu_t^{\mathrm{RS}}}(i)\Big)$$

represents the degree at t of the i^{th} susceptible individual (resp. the number of edges to S of the i^{th} infectious individual and of the i^{th} removed individual) when individuals are ranked by increasing degrees (resp. by number of edges to S).

Example 1. Consider for instance the measure $\mu = 2\delta_1 + 3\delta_5 + \delta_7$. Then, the atoms 1 and 2 are at level 1, the atoms 3, 4 and 5 are at level 5, and the atom 6 is at level 7. We then have that $\gamma_1(\mu) = F_\mu^{-1}(1) = 1$, $\gamma_2(\mu) = 1$, $\gamma_3(\mu) = \gamma_4(\mu) = \gamma_5(\mu) = 5$, and $\gamma_6(\mu) = 7$. □

From t, and because of the properties of exponential distributions the next event will take place in a time exponentially distributed with parameter $rN_t^{\mathrm{IS}} + \beta I_t$. Let T denote the time of this event.

Case 1 The event corresponds to a removal, i.e., a node goes from status I to status R. Choose uniformly an integer i in I_{T-}, then update the measures $\mu_{T_-}^{\mathrm{IS}}$ and $\mu_{T_-}^{\mathrm{RS}}$:

$$\mu_T^{\mathrm{IS}} = \mu_{T-}^{\mathrm{IS}} - \delta_{\gamma_i(\mu_{T_-}^{\mathrm{IS}})} \text{ and } \mu_T^{\mathrm{RS}} = \mu_{T-}^{\mathrm{RS}} + \delta_{\gamma_i(\mu_{T_-}^{\mathrm{IS}})}.$$

The probability that a given integer i is drawn is $1/I_{T-}$.

Case 2 The event corresponds to a new infection. We choose uniformly a half-edge with susceptible ego, and this ego becomes infectious. The global rate of infection is $rN_{T_-}^{\mathrm{IS}}$ and the probability of choosing a susceptible individual of degree k for the new infectious is $k\mu_{T_-}^{\mathrm{S}}(k)/N_{T_-}^{\mathrm{S}}$. We define by

$$\lambda_{T_-}(k) = rk\frac{N_s^{\mathrm{IS}}}{N_s^{\mathrm{S}}} \tag{2.1}$$

the rate of infection of a *given* susceptible of degree k at time T_-. This notation was also used in Volz [27].

The newly infective may have several links with infectious or removed individuals. The probability, given that the degree of the individual is k and that j (resp. ℓ) out of her $k-1$ other half-edges (all but the contaminating IS edge) are chosen to be of type II (resp. IR), according to Step 2', is given by the following multivariate hypergeometric distribution:

$$p_{T_-}(j, \ell \,|\, k-1) = \frac{\binom{N_{T_-}^{\mathrm{IS}}-1}{j}\binom{N_{T_-}^{\mathrm{RS}}}{\ell}\binom{N_{T_-}^{\mathrm{S}}-N_{T_-}^{\mathrm{IS}}-N_{T_-}^{\mathrm{RS}}}{k-1-j-\ell}}{\binom{N_{T_-}^{\mathrm{S}}-1}{k-1}}. \tag{2.2}$$

Finally, to update the values of μ_T^{IS} and μ_T^{RS} given k, j and ℓ, we have to choose the infectious and removed individuals to which the newly infectious is linked: some of their edges, which were IS or RS, now become II or RI. We draw two sequences $u = (u_1, \dots, u_{I_{T_-}})$ and $v = (v_1, \dots, v_{R_{T_-}})$ that will indicate how many links each infectious or removed individual has to the newly contaminated individual. There exist constraints on u and v: the number of edges recorded by u and v can not exceed the number of existing edges. Let us define the set

$$\mathcal{U} = \bigcup_{m=1}^{+\infty} \mathbb{N}^m, \tag{2.3}$$

and for all finite integer-valued measure μ on $\mathbb{N}$, and all integer $n \in \mathbb{N}$, we define the subset

$$\mathcal{U}(n,\mu) = \Big\{ u = (u_1, ..., u_{\langle \mu, \mathbf{1}\rangle}) \in \mathbb{N}^{\langle \mu, \mathbf{1}\rangle} \quad \text{such that} \\ \forall i \in \{1, \dots, \langle \mu, \mathbf{1}\rangle\},\ u_i \leq F_\mu^{-1}(i) \text{ and } \sum_{i=1}^{\langle \mu, \mathbf{1}\rangle} u_i = n \Big\}. \tag{2.4}$$

Each sequence $u \in \mathcal{U}(n,\mu)$ provides a possible configuration of how the n connections of a given individual can be shared between neighbors whose degrees are summed up by μ. The component u_i, for $1 \leq i \leq \langle \mu, 1\rangle$, provides the number of edges that this individual shares with the ith individual. This number is necessarily smaller than the degree $\gamma_i(\mu) = F_\mu^{-1}(i)$ of individual i. Moreover, the u_i's sum to n. The probabilities of the draws of u and v that provide respectively the number of edges IS which become II per infectious individual and the number of edges RS which become RI per removed individual are given by:

$$\begin{aligned} \rho(u|j+1, \mu^{\text{IS}}_{T_-}) &= \frac{\prod_{i=1}^{I_{T_-}} \binom{\gamma_i(\mu^{\text{IS}}_{T_-})}{u_i}}{\binom{N^{\text{IS}}_{T_-}}{j+1}} \mathbb{1}_{u \in \mathcal{U}(j+1, \mu^{\text{IS}}_{T_-})} \\ \rho(v|\ell, \mu^{\text{RS}}_{T_-}) &= \frac{\prod_{i=1}^{R_{T_-}} \binom{\gamma_i(\mu^{\text{RS}}_{T_-})}{v_i}}{\binom{N^{\text{RS}}_{T_-}}{\ell}} \mathbb{1}_{v \in \mathcal{U}(\ell, \mu^{\text{RS}}_{T_-})}. \end{aligned} \tag{2.5}$$

Then, we then update the measures as follows:

$$\begin{aligned} \mu^{\text{S}}_T &= \mu^{\text{S}}_{T-} - \delta_k \\ \mu^{\text{IS}}_T &= \mu^{\text{IS}}_{T-} + \delta_{k-1-j-\ell} + \sum_{i=1}^{\text{I}_{T_-}} \delta_{\gamma_i(\mu^{\text{IS}}_{T_-}) - u_i} - \delta_{\gamma_i(\mu^{\text{IS}}_{T_-})} \\ \mu^{\text{RS}}_T &= \mu^{\text{RS}}_{T-} + \sum_{i'=1}^{\text{R}_{T_-}} \delta_{\gamma_{i'}(\mu^{\text{RS}}_{T_-}) - v_{i'}} - \delta_{\gamma_{i'}(\mu^{\text{RS}}_{T_-})}. \end{aligned} \tag{2.6}$$

2.3 Stochastic differential equations

Here, we propose stochastic differential equations (SDEs) driven by Poisson point measures (PPMs) to describe the evolution of the degree distributions (1.1), following the inspiration of [9, 13]. We consider two PPMs: $\mathrm{d}Q^1(s,k,\theta_1,j,\ell,\theta_2,u,\theta_3,v,\theta_4)$ and $\mathrm{d}Q^2(s,i)$ on $\mathbb{R}_+ \times E_1$ with $E_1 := \mathbb{N} \times \mathbb{R}_+ \times \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+ \times \mathcal{U} \times \mathbb{R}_+ \times \mathcal{U} \times \mathbb{R}_+$ and $\mathbb{R}_+ \times \mathbb{N}$ with intensity measures $\mathrm{d}q^1(s,k,\theta_1,j,\ell,\theta_2,u,\theta_3,v,\theta_4) = \mathrm{d}s \otimes \mathrm{d}n(k) \otimes \mathrm{d}\theta_1 \otimes \mathrm{d}n(j) \otimes \mathrm{d}n(\ell) \otimes \mathrm{d}\theta_2 \otimes \mathrm{d}n(u) \otimes \mathrm{d}\theta_3 \otimes \mathrm{d}n(v) \otimes \mathrm{d}\theta_4$ and $\mathrm{d}q^2(s,i) = \beta\ \mathrm{d}s \otimes \mathrm{d}n(i)$, where $\mathrm{d}s$, $\mathrm{d}\theta_1$, $\mathrm{d}\theta_2$, $\mathrm{d}\theta_3$ and $\mathrm{d}\theta_4$ are Lebesgue measures on $\mathbb{R}_+$, where $\mathrm{d}n(k)$, $\mathrm{d}n(j)$, $\mathrm{d}n(\ell)$ are counting measures on $\mathbb{N}$, and where $\mathrm{d}n(u)$, $\mathrm{d}n(v)$ are counting measures on $\mathcal{U}$.

The point measure Q^1 provides possible times at which an infection may occur. Each of its atoms is associated with a possible infection time s, an integer k which gives the degree of the susceptible being possibly infected, the number $j+1$ and ℓ of edges that this individual has to the sets I_{s_-} and R_{s_-}. The marks u and $v \in \mathcal{U}$ are as in the previous section. The marks θ_1, θ_2 and θ_3 are auxiliary variables used for the construction (see (2.8)-(2.9)).
The point measure Q^2 gives possible removal times. To each of its atoms is associated a possible removal time s and the number i of the individual that may be removed.

The following SDEs describe the evolution of the epidemic: for all $t \geq 0$,

$$\mu_t^{\mathrm{S}} = \mu_0^{\mathrm{S}} - \int_0^t \int_{E_1} \delta_k \mathbb{1}_{\theta_1 \leq \lambda_{s\_}(k)\mu_{s\_}^{\mathrm{S}}(k)} \mathbb{1}_{\theta_2 \leq p_{s\_}(j,\ell|k-1)} \mathbb{1}_{\theta_3 \leq \rho(u|j+1,\mu_{s\_}^{\mathrm{IS}})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell,\mu_{s\_}^{\mathrm{RS}})} \, \mathrm{d}Q^1 \tag{2.7}$$

$$\begin{aligned} \mu_t^{\mathrm{IS}} = \mu_0^{\mathrm{IS}} + \int_0^t \int_{E_1} & \Big(\delta_{k-(j+1+\ell)} + \sum_{i=1}^{I_{s\_}} \big(\delta_{\gamma_i(\mu_{s\_}^{\mathrm{IS}})-u_i} - \delta_{\gamma_i(\mu_{s\_}^{\mathrm{IS}})}\big)\Big) \\ & \times \mathbb{1}_{\theta_1 \leq \lambda_{s\_}(k)\mu_{s\_}^{\mathrm{S}}(k)} \mathbb{1}_{\theta_2 \leq p_{s\_}(j,\ell|k-1)} \mathbb{1}_{\theta_3 \leq \rho(u|j+1,\mu_{s\_}^{\mathrm{IS}})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell,\mu_{s\_}^{\mathrm{RS}})} \, \mathrm{d}Q^1 \\ & - \int_0^t \int_{\mathbb{N}} \delta_{\gamma_i(\mu_{s\_}^{\mathrm{IS}})} \mathbb{1}_{i \leq I_{s\_}} \, \mathrm{d}Q^2 \end{aligned} \tag{2.8}$$

$$\begin{aligned} \mu_t^{\mathrm{RS}} = \mu_0^{\mathrm{RS}} + \int_0^t \int_{E_1} & \Big(\sum_{i=1}^{R_{s\_}} \big(\delta_{\gamma_i(\mu_{s\_}^{\mathrm{RS}})-v_i} - \delta_{\gamma_i(\mu_{s\_}^{\mathrm{RS}})}\big)\Big) \\ & \times \mathbb{1}_{\theta_1 \leq \lambda_{s\_}(k)\mu_{s\_}^{\mathrm{S}}(k)} \mathbb{1}_{\theta_2 \leq p_{s\_}(j,\ell|k-1)} \mathbb{1}_{\theta_3 \leq \rho(u|j+1,\mu_{s\_}^{\mathrm{IS}})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell,\mu_{s\_}^{\mathrm{RS}})} \, \mathrm{d}Q^1 \\ & + \int_0^t \int_{\mathbb{N}} \delta_{\gamma_i(\mu_{s\_}^{\mathrm{IS}})} \mathbb{1}_{i \leq I_{s\_}} \, \mathrm{d}Q^2, \end{aligned} \tag{2.9}$$

where we write $\mathrm{d}Q^1$ and $\mathrm{d}Q^2$ instead of $\mathrm{d}Q^1(s,k,\theta_1,j,\ell,\theta_2,u,\theta_3,v,\theta_4)$ and $\mathrm{d}Q^2(s,i)$ to simplify the notation.

Proposition 2.1. *For any given initial conditions μ_0^{S}, μ_0^{SI} and μ_0^{RS} that are integer-valued measures on $\mathbb{N}$ and for PPMs Q^1 and Q^2, there exists a unique strong solution to the SDEs (2.7)-(2.9) in the space $\mathcal{D}\big(\mathbb{R}_+, (\mathcal{M}_F(\mathbb{N}))^3\big)$, the Skohorod space of càdlàg functions with values in $(\mathcal{M}_F(\mathbb{N}))^3$.*

Proof. For the proof, we notice that for every $t \in \mathbb{R}_+$, the measure μ_t^{S} is dominated by μ_0^{S} and the measures μ_t^{IS} and μ_t^{RS} have a mass bounded by $\langle \mu_0^{\mathrm{S}} + \mu_0^{\mathrm{IS}} + \mu_0^{\mathrm{RS}}, 1\rangle$ and a support included in $[\![0, \max\{\max(\text{supp } (\mu_0^{\mathrm{S}})), \max(\text{supp } (\mu_0^{\mathrm{IS}})), \max(\text{supp } (\mu_0^{\mathrm{RS}}))\}]\!]$. The result then follows the steps of [13] and [25] (Proposition 2.2.6). ■

The course of the epidemic can be deduced from (2.7), (2.8) and (2.9). For the sizes $(S_t, I_t, R_t)_{t\in\mathbb{R}_+}$ of the different classes, for instance, we have with the choice of $f \equiv 1$ that for all $t \geq 0$, $S_t = \langle \mu_t^{\mathrm{S}}, \mathbf{1}\rangle$, $I_t = \langle \mu_t^{\mathrm{IS}}, \mathbf{1}\rangle$ and $R_t = \langle \mu_t^{\mathrm{RS}}, \mathbf{1}\rangle$. Writing the semi-martingale decomposition that results from standard stochastic calculus for jump processes and SDE driven by PPMs (*e.g.* [13, 14, 15]), we obtain for example:

$$I_t = \langle \mu_t^{\mathrm{IS}}, \mathbf{1}\rangle = I_0 + \int_0^t \Big(\sum_{k\in\mathbb{N}} \mu_s^{\mathrm{S}}(k)\lambda_s(k) - \beta\, I_s\Big)\, \mathrm{d}s + M_t^{\mathrm{I}}, \tag{2.10}$$

where M^{I} is a square-integrable martingale that can be written explicitly with the compensated PPMs of Q^1 and Q^2, and with predictable quadratic variation given for all $t \geq 0$ by

$$\langle M^{\mathrm{I}}\rangle_t = \int_0^t \sum_{k\in\mathbb{N}} \Big(\mu_s^{\mathrm{S}}(k)\lambda_s(k) + \beta I_s\Big)\, \mathrm{d}s.$$

Another quantities of interest are the numbers of edges of the different types N_t^{S}, N_t^{IS}, N_t^{RS}. The latter appear as the first moments of the measures μ_t^{S}, μ_t^{IS} and μ_t^{RS}: $N_t^{\mathrm{S}} = \langle \mu_t^{\mathrm{S}}, \chi\rangle$, $N_t^{\mathrm{IS}} = \langle \mu_t^{\mathrm{IS}}, \chi\rangle$ and $N_t^{\mathrm{RS}} = \langle \mu_t^{\mathrm{RS}}, \chi\rangle$.

3 Large graph limit

Volz [27] proposed a parsimonious deterministic approximation to describe the epidemic dynamics when the population is large. However, the stochastic processes are not clearly defined and the convergence of the SDEs to the 4 ODEs that Volz proposes is stated but not proved. Using the construction that we developed in Section 2.2, we provide mathematical proofs of Volz' equation, starting from a finite graph and taking the limit when the size of the graph tends to infinity. Moreover, we see that the three distributions μ^{S}, μ^{IS} and μ^{RS} are at the core of the problem and encapsulate the evolution of the process.

3.1 Law of Large Numbers scaling

We consider sequences of measures $(\mu^{n,\text{S}})_{n\in\mathbb{N}}$, $(\mu^{n,\text{IS}})_{n\in\mathbb{N}}$ and $(\mu^{n,\text{RS}})_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}^*$, $\mu^{n,\text{S}}$, $\mu^{n,\text{IS}}$ and $\mu^{n,\text{RS}}$ satisfy (2.7)-(2.9) with initial conditions $\mu_0^{n,\text{S}}$, $\mu_0^{n,\text{IS}}$ and $\mu_0^{n,\text{RS}}$. We denote by S_t^n, I_t^n and R_t^n the subclasses of susceptible, infectious or removed individuals at time t, and by $N_t^{n,\text{S}}$, $N_t^{n,\text{IS}}$ and $N_t^{n,\text{RS}}$, the number of edges with susceptible ego, infectious ego and susceptible alter, removed ego and susceptible alter. The number of vertices of each class are denoted I_t^n, S_t^n and R_t^n. The total size of the population is finite and equal to $S_0^n + I_0^n + R_0^n$. The size of the population and the number of edges tend to infinity proportionally to n.

We scale the measures the following way. For any $n \geq 0$, we set

$$\mu_t^{(n),\text{IS}} = \frac{1}{n}\mu_t^{n,\text{IS}}$$

for all $t \geq 0$ (and accordingly, $\mu_t^{(n),\text{S}}$ and $\mu_t^{(n),\text{RS}}$). Then, we denote

$$N_t^{(n),\text{IS}} = \langle \mu^{(n),\text{IS}}, \chi \rangle = \frac{1}{n} N_t^{n,\text{IS}}, \quad \text{ and } \quad I_t^{(n)} = \langle \mu_t^{(n),\text{IS}}, \mathbf{1} \rangle = \frac{1}{n} I_t^n$$

and accordingly, $N_t^{(n),\text{S}}$, $N_t^{(n),\text{RS}}$, $S_t^{(n)}$ and $R_t^{(n)}$.

We assume that the initial conditions satisfy:

Assumption 3.1. *The sequences $(\mu_0^{(n),\text{S}})_{n\in\mathbb{N}}$, $(\mu_0^{(n),\text{IS}})_{n\in\mathbb{N}}$ and $(\mu_0^{(n),\text{RS}})_{n\in\mathbb{N}}$ converge to measures $\bar{\mu}_0^{\text{S}}$, $\bar{\mu}_0^{\text{IS}}$ and $\bar{\mu}_0^{\text{RS}}$ in $\mathcal{M}_F(\mathbb{N})$ embedded with the weak convergence topology.*

Remark 1. 1. Assumption 3.1 entails that the initial (susceptible and infectious) population size is of order n if $\bar{\mu}_0^{\text{S}}$ and $\bar{\mu}_0^{\text{IS}}$ are nontrivial.
2. In case the distributions underlying the measures $\mu_0^{n,\text{S}}$, $\mu_0^{n,\text{IS}}$ and $\mu_0^{n,\text{RS}}$ do not depend on the total number of vertices (*e.g.* Poisson, power-laws or geometric distributions), Assumption 3.1 can be viewed as a law of large numbers. When the distributions depend on the total number of vertices N (as in Erdös-Renyi graphs), there may be scalings under which Assumption 3.1 holds. For Erdös-Renyi graphs for instance, if the probability ρ_N of connecting two vertices satisfies $\lim_{N\to+\infty} N\rho_N = \lambda$, then we obtain in the limit a Poisson distribution with parameter λ.
3. In Equation (1.4), notice the appearance of the size biased degree distribution $k\bar{\mu}_s^{\text{S}}(k)/N_s^{\text{S}}$. The latter reflects the fact that, in the CM, individuals having large degrees have higher probability to connect than individuals having small degrees. Thus, there is no reason why the degree distributions of the susceptible individuals $\bar{\mu}_0^{\text{S}}/\bar{S}_0$ and the distribution $\sum_{k\in\mathbb{N}} p_k\delta_k$ underlying the CM should coincide. Assumption 3.1 tells us indeed that the initial infectious population size is of order n. Even if $\bar{I}_0/\bar{S}_0$ is very small, the biased distributions that appear imply that the degree distribution $\bar{\mu}_0^{\text{IS}}/\bar{I}_0$ should have a larger expectation than the degree distribution $\bar{\mu}_0^{\text{S}}/\bar{S}_0$. $\square$

We obtain rescaled SDEs which are the same as the SDEs (2.7)-(2.9) parameterized by n. For all $t \geq 0$

$$\mu_t^{(n),\mathrm{S}} = \mu_0^{(n),\mathrm{S}} - \frac{1}{n}\int_0^t \int_{E_1} \delta_k \mathbb{1}_{\theta_1 \leq \lambda^n_{s\_}(k) n\mu_{s\_}^{(n),\mathrm{S}}(k)} \mathbb{1}_{\theta_2 \leq p^n_{s\_}(j,\ell|k-1)} \times \mathbb{1}_{\theta_3 \leq \rho(u|j+1, n\mu_{s\_}^{(n),\mathrm{IS}})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell, n\mu_{s\_}^{(n),\mathrm{RS}})} \, \mathrm{d}Q^1, \tag{3.1}$$

$$\begin{aligned}\mu_t^{(n),\mathrm{IS}} = \mu_0^{(n),\mathrm{IS}} + \frac{1}{n}\int_0^t \int_{E_1} \Big(\delta_{k-(j+1+\ell)} + \sum_{i=1}^{I^n_{s\_}} \big(\delta_{\gamma_i(n\mu_{s\_}^{(n),\mathrm{IS}})-u_i} - \delta_{\gamma_i(n\mu_{s\_}^{(n),\mathrm{IS}})}\big)\Big) \\ \times \mathbb{1}_{\theta_1 \leq \lambda^n_{s\_}(k) n\mu_{s\_}^{(n),\mathrm{S}}(k)} \mathbb{1}_{\theta_2 \leq p^n_{s\_}(j,\ell|k-1)} \mathbb{1}_{\theta_3 \leq \rho(u|j+1, n\mu_{s\_}^{(n),\mathrm{IS}})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell, n\mu_{s\_}^{(n),\mathrm{RS}})} \, \mathrm{d}Q^1 \\ - \frac{1}{n}\int_0^t \int_{\mathbb{N}} \delta_{\gamma_i(n\mu_{s\_}^{(n),\mathrm{IS}})} \mathbb{1}_{i \in \mathrm{I}^n_{s\_}} \, \mathrm{d}Q^2,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\mu_t^{(n),\mathrm{RS}} = \mu_0^{(n),\mathrm{RS}} + \frac{1}{n}\int_0^t \int_{E_1} \Big(\sum_{i=1}^{R^n_{s\_}} \big(\delta_{\gamma_i(n\mu_{s\_}^{(n),\mathrm{RS}})-v_i} - \delta_{\gamma_i(n\mu_{s\_}^{(n),\mathrm{RS}})}\big)\Big) \\ \times \mathbb{1}_{\theta_1 \leq \lambda^n_{s\_}(k) n\mu_{s\_}^{(n),\mathrm{S}}(k)} \mathbb{1}_{\theta_2 \leq p^n_{s\_}(j,\ell|k-1)} \mathbb{1}_{\theta_3 \leq \rho(u|j+1, n\mu_{s\_}^{(n),\mathrm{IS}})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell, n\mu_{s\_}^{(n),\mathrm{RS}})} \, \mathrm{d}Q^1 \\ + \frac{1}{n}\int_0^t \int_{\mathbb{N}} \delta_{\gamma_i(n\mu_{s\_}^{(n),\mathrm{IS}})} \mathbb{1}_{i \in \mathrm{I}^n_{s\_}} \, \mathrm{d}Q^2,\end{aligned} \tag{3.3}$$

where we denote for all $s \geq 0$

$$\lambda^n_s(k) = rk\frac{N^{n,\mathrm{IS}}_s}{N^{n,\mathrm{S}}_s}, \text{ and } p^n_s(j,\ell \mid k-1) = \frac{\binom{N^{n,\mathrm{IS}}_s - 1}{j}\binom{N^{n,\mathrm{RS}}_s}{\ell}\binom{N^{n,\mathrm{S}}_s - N^{n,\mathrm{IS}}_s - N^{n,\mathrm{RS}}_s}{k-1-j-\ell}}{\binom{N^{n,\mathrm{S}}_s - 1}{k-1}}. \tag{3.4}$$

Several semi-martingale decompositions will be useful in the sequel. We focus on $\mu^{(n),\mathrm{IS}}$ but similar decompositions hold for $\mu^{(n),\mathrm{S}}$ and $\mu^{(n),\mathrm{RS}}$, which we do not detail since they can be deduced by direct adaptation of the following computation.

Proposition 3.2. *For all $f \in \mathcal{B}_b(\mathbb{N})$, for all $t \geq 0$,*

$$\langle \mu_t^{(n),\mathrm{IS}}, f\rangle = \sum_{k\in\mathbb{N}} f(k)\mu_0^{(n),\mathrm{IS}}(k) + A_t^{(n),\mathrm{IS},f} + M_t^{(n),\mathrm{IS},f}, \tag{3.5}$$

where the finite variation part $A_t^{(n),\mathrm{IS},f}$ of $\langle \mu_t^{(n),\mathrm{IS}}, f\rangle$ reads

$$\begin{aligned}A_t^{(n),\mathrm{IS},f} = \int_0^t \sum_{k\in\mathbb{N}} \lambda^n_s(k)\mu_s^{(n),\mathrm{S}}(k) \sum_{j+\ell+1\leq k} p^n_s(j,\ell|k-1) \sum_{u\in\mathcal{U}} \rho(u|j+1, \mu_s^{n,\mathrm{IS}}) \\ \times \Big(f(k-(j+1+\ell)) + \sum_{i=1}^{I^n_s} \big(f(\gamma_i(\mu_s^{n,\mathrm{IS}}) - u_i) - f(\gamma_i(\mu_s^{n,\mathrm{IS}}))\big)\Big) \, ds \\ - \int_0^t \beta\langle \mu_s^{(n),\mathrm{IS}}, f\rangle \, ds,\end{aligned} \tag{3.6}$$

and where the martingale part $M_t^{(n),\mathrm{IS},f}$ of $\langle \mu_t^{(n),\mathrm{IS}}, f\rangle$ is a square integrable martingale starting *from 0 with quadratic variation*

$$\begin{aligned}\langle M^{(n),\text{IS},f}\rangle_t =& \frac{1}{n}\int_0^t \beta\langle \mu_s^{(n),\text{IS}}, f^2\rangle \; ds \\ &+ \frac{1}{n}\int_0^t \sum_{k\in\mathbb{N}} \lambda_s^n(k)\mu_s^{(n),\text{S}}(k) \sum_{j+\ell+1\le k} p_s^n(j,\ell|k-1)\sum_{u\in\mathcal{U}} \rho(u|j+1,\mu_s^{n,\text{IS}}) \\ &\qquad \times \Big(f\left(k-(j+1+\ell)\right) + \sum_{i=1}^{I_s^n} \left(f\left(\gamma_i(\mu_s^{n,\text{IS}})-u_i\right) - f\left(\gamma_i(\mu_s^{n,\text{IS}})\right)\right)\Big)^2 \; ds.\end{aligned}$$

Proof. The proof proceeds from (3.2) and standard stochastic calculus for jump processes (see *e.g.* [13]). ■

3.2 Convergence of the normalized process

We aim to study the limit of the system when $n \to +\infty$. We introduce the associated measure spaces. For any $\varepsilon \ge 0$ and $A > 0$, we define the following closed set of $\mathcal{M}_F(\mathbb{N})$ as

$$\mathcal{M}_{\varepsilon,\,A} = \{\nu \in \mathcal{M}_F(\mathbb{N}) \; ; \; \langle \nu, \mathbf{1}+\chi^5\rangle \le A \text{ and } \langle \nu,\, \chi\rangle \ge \varepsilon\} \tag{3.7}$$

and $\mathcal{M}_{0+,A} = \cup_{\varepsilon>0}\mathcal{M}_{\varepsilon,A}$. Topological properties of these spaces are given in Appendix A.

In the proof, we will see that the epidemic remains large provided the number of edges from I to S remains of the order of n. Let us thus define, for all $\varepsilon > 0$, $\varepsilon' > 0$ and $n \in \mathbb{N}^*$,

$$t_{\varepsilon'} := \inf\{t \ge 0, \langle \bar{\mu}_t^{\text{IS}}, \chi\rangle < \varepsilon'\} \tag{3.8}$$

and:

$$\tau_\varepsilon^n = \inf\{t \ge 0,\, \langle \mu_t^{(n),\text{IS}}, \chi\rangle < \varepsilon\}. \tag{3.9}$$

Our main result is the following Theorem.

Theorem 1. *Suppose that Assumption 3.1 holds with*

$$\left(\mu_0^{(n),\text{S}},\, \mu_0^{(n),\text{IS}},\, \mu_0^{(n),\text{RS}}\right) \; in \; (\mathcal{M}_{0,A})^3 \; for \; any \; n, \;\; with \; \langle \bar{\mu}_0^{\text{IS}}, \chi\rangle > 0. \tag{3.10}$$

Then, we have:
1. there exists a unique solution $(\bar{\mu}^{\text{S}}, \bar{\mu}^{\text{IS}}, \bar{\mu}^{\text{RS}})$ to the deterministic system of measure-valued equations (1.3)-(1.5) in $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_{0,A}\times\mathcal{M}_{0+,A}\times\mathcal{M}_{0,A})$,
2. when n converges to infinity, the sequence $(\mu^{(n),\text{S}}, \mu^{(n),\text{IS}}, \mu^{(n),\text{RS}})_{n\in\mathbb{N}}$ converges in distribution in $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{0,A}^3)$ to $(\bar{\mu}^{\text{S}}, \bar{\mu}^{\text{IS}}, \bar{\mu}^{\text{RS}})$.

Proof of Theorem 1. Uniqueness of the solution to (1.3)-(1.5) is proved in Step 2. For the proof of (2), since $\lim_{\varepsilon'\to 0} t_{\varepsilon'} = +\infty$, it is sufficient to prove the results on $\mathcal{D}([0,t_{\varepsilon'}], \mathcal{M}_{0,A}^3)$ for ε' small enough. In the sequel, we choose $0 < \varepsilon < \varepsilon' < \langle \bar{\mu}_0^{\text{IS}}, \chi\rangle$.
Step 1 Let us prove that $(\mu^{(n),\text{S}}, \mu^{(n),\text{IS}}, \mu^{(n),\text{RS}})_{n\in\mathbb{N}^*}$ is tight. Let $t \in \mathbb{R}_+$ and $n \in \mathbb{N}^*$. By hypothesis, we have that

$$\begin{aligned}\langle \mu_t^{(n),\text{S}}, \mathbf{1}+\chi^5\rangle + \langle \mu_t^{(n),\text{IS}}, \mathbf{1}+\chi^5\rangle + \langle \mu_t^{(n),\text{RS}}, \mathbf{1}+\chi^5\rangle& \\ \le \langle \mu_0^{(n),\text{S}}, \mathbf{1}+\chi^5\rangle + \langle \mu_0^{(n),\text{IS}}, \mathbf{1}+\chi^5\rangle \le 2A.&\end{aligned} \tag{3.11}$$

Thus the sequences $(\mu_t^{(n),\text{S}})_{n\in\mathbb{N}^*}$, $(\mu_t^{(n),\text{IS}})_{n\in\mathbb{N}^*}$ and $(\mu_t^{(n),\text{RS}})_{n\in\mathbb{N}^*}$ are tight for each $t \in \mathbb{R}_+$. Now by the criterion of Roelly [24], it remains to prove that for each $f \in \mathcal{B}_b(\mathbb{N})$, the sequence

$(\langle\mu^{(n),\text{S}}_{\cdot},f\rangle,\langle\mu^{(n),\text{IS}}_{\cdot},f\rangle,\langle\mu^{(n),\text{RS}}_{\cdot},f\rangle)_{n\in\mathbb{N}^*}$ is tight in $\mathcal{D}(\mathbb{R}_+,\mathbb{R}^3)$. Since we have semi-martingale decompositions of these processes, it is sufficient, by using the Rebolledo criterion, to prove that the finite variation part and the bracket of the martingale satisfy the Aldous criterion (see *e.g.* [16]). We only prove that $\langle\mu^{(n),\text{IS}}_{\cdot},f\rangle$ is tight. For the other components, the computations are similar.

The Rebolledo-Aldous criterion is satisfied if for all $\alpha>0$ and $\eta>0$ there exists $n_0\in\mathbb{N}$ and $\delta>0$ such that for all $n>n_0$ and for all stopping times S_n and T_n such that $S_n<T_n<S_n+\delta$,

$$\begin{aligned}&\mathbb{P}\big(|A^{(n),\text{IS},f}_{T_n}-A^{(n),\text{IS},f}_{S_n}|>\eta\big)\leq\alpha,\quad\text{ and}\\&\mathbb{P}\big(|\langle M^{(n),\text{IS},f}\rangle_{T_n}-\langle M^{(n),\text{IS},f}\rangle_{S_n}|>\eta\big)\leq\alpha.\end{aligned}\tag{3.12}$$

For the finite variation part,

$$\begin{aligned}\mathbb{E}\Big[|A^{(n),\text{IS},f}_{T_n}-A^{(n),\text{IS},f}_{S_n}|\Big]\leq{}&\mathbb{E}\left[\int_{S_n}^{T_n}\beta\|f\|_\infty\langle\mu^{(n),\text{IS}}_s,1\rangle\ \mathrm{d}s\right]\\&+\mathbb{E}\left[\int_{S_n}^{T_n}\sum_{k\in\mathbb{N}}\lambda^n_s(k)\mu^{(n),\text{S}}_s(k)\sum_{j+\ell\leq k-1}p^n_s(j,\ell|k-1)(2j+3)\|f\|_\infty\ \mathrm{d}s\right]\end{aligned}$$

The term $\sum_{j+\ell\leq k-1}jp^n_s(j,\ell|k-1)$ is the mean number of links to $\text{I}^n_{s\_}$ that the newly infected individual has, given that this individual is of degree k. It is bounded by k. Then, with (3.4),

$$\mathbb{E}\Big[|A^{(n),\text{IS},f}_{T_n}-A^{(n),\text{IS},f}_{S_n}|\Big]\leq\delta\mathbb{E}\Big[\beta\|f\|_\infty(S^{(n)}_0+I^{(n)}_0)+r\|f\|_\infty\langle\mu^{(n),\text{S}}_0,2\chi^2+3\chi\rangle\Big],$$

by using that the number of infectives is bounded by the size of the population and that $\mu^{(n),\text{S}}_s(k)\leq\mu^{(n),\text{S}}_0(k)$ for all k and $s\geq0$. From (3.10), the r.h.s. is finite. Using Markov's inequality,

$$\mathbb{P}\big(|A^{(n),\text{IS},f}_{T_n}-A^{(n),\text{IS},f}_{S_n}|>\eta\big)\leq\frac{(5r+2\beta)A\delta\|f\|_\infty}{\eta},$$

which is smaller than α for δ small enough.

We use the same arguments for the bracket of the martingale:

$$\begin{aligned}&\mathbb{E}\big[|\langle M^{(n),\text{IS},f}\rangle_{T_n}-\langle M^{(n),\text{IS},f}\rangle_{S_n}|\big]\\&\qquad\leq\mathbb{E}\Big[\frac{\delta\beta\|f\|^2_\infty(S^{(n)}_0+I^{(n)}_0)}{n}+\frac{\delta r\|f\|^2_\infty\langle\mu^{(n),\text{S}}_0,\chi(2\chi+3)^2\rangle}{n}\Big]\\&\qquad\leq\frac{(25r+2\beta)A\delta\|f\|^2_\infty}{n},\end{aligned}\tag{3.13}$$

using Assumption (3.10). The r.h.s. can be made smaller than $\eta\alpha$ for a small enough δ, so the second inequality of (3.12) follows again from Markov's inequality. By [24], this provides the tightness in $\mathcal{D}(\mathbb{R}_+,\mathcal{M}^3_{0,A})$.

By Prohorov theorem (*e.g.* [11], p.104) and Step 1, the distributions of $(\mu^{(n),\text{S}},\mu^{(n),\text{IS}},\mu^{(n),\text{RS}})$, for $n\in\mathbb{N}^*$, form a relatively compact family of bounded measures on $\mathcal{D}(\mathbb{R}_+,\mathcal{M}^3_{0,A})$, and so do the laws of the stopped processes $(\mu^{(n),\text{S}}_{\cdot\wedge\tau^n_\varepsilon},\mu^{(n),\text{IS}}_{\cdot\wedge\tau^n_\varepsilon},\mu^{(n),\text{RS}}_{\cdot\wedge\tau^n_\varepsilon})_{n\in\mathbb{N}^*}$ (recall (3.9)). Let $\bar\mu:=(\bar\mu^{\text{S}},\bar\mu^{\text{IS}},\bar\mu^{\text{RS}})$ be a limiting point in $\mathcal{C}(\mathbb{R}_+,\mathcal{M}^3_{0,A})$ of the sequence of stopped processes and let us consider a subsequence again denoted by $\mu^{(n)}:=(\mu^{(n),\text{S}},\mu^{(n),\text{IS}},\mu^{(n),\text{RS}})_{n\in\mathbb{N}^*}$, with an abuse of notation, and that converges to $\bar\mu$. Because the limiting values are continuous, the convergence of $(\mu^{(n)})_{n\in\mathbb{N}^*}$ to $\bar\mu$ holds for the uniform convergence on every compact subset of $\mathbb{R}_+$ (*e.g.* [6] p.112).

Now, let us define for all $t \in \mathbb{R}_+$ and for all bounded function f on $\mathbb{N}$, the mappings $\Psi_t^{\text{S},f}$, $\Psi_t^{\text{IS},f}$ and $\Psi_t^{\text{RS},f}$ from $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{0,A}^3)$ into $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ such that (1.3)-(1.5) read

$$(\langle \bar{\mu}_t^{\text{S}}, f\rangle, \langle \bar{\mu}_t^{\text{IS}}, f\rangle, \langle \bar{\mu}_t^{\text{RS}}, f\rangle) = \Big(\Psi_t^{\text{S},f}\left(\bar{\mu}^{\text{S}}, \bar{\mu}^{\text{IS}}, \bar{\mu}^{\text{RS}}\right), \Psi_t^{\text{IS},f}\left(\bar{\mu}^{\text{S}}, \bar{\mu}^{\text{IS}}, \bar{\mu}^{\text{RS}}\right), \Psi_t^{\text{RS},f}\left(\bar{\mu}^{\text{S}}, \bar{\mu}^{\text{IS}}, \bar{\mu}^{\text{RS}}\right)\Big). \quad (3.14)$$

Our purpose is to prove that the limiting values are the unique solution of Equations (1.3)-(1.5). Before proceeding to the proof, a remark is in order. A natural way of reasoning would be to prove that $\Psi^{\text{S},f}$, $\Psi^{\text{IS},f}$ and $\Psi^{\text{RS},f}$ are Lipschitz continuous in some spaces of measures. It turns that this can only be done by considering the set of measures with moments of any order, which is a set too small for applications. We circumvent this difficulty by first proving that the mass and the first two moments of any solutions of the system are the same. Then, we prove that the generating functions of these measures satisfy a partial differential equation known to have a unique solution.

Step 2 We now prove that the differential system (1.3)-(1.5) has at most one solution in $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_{0,A} \times \mathcal{M}_{0+,A} \times \mathcal{M}_{0,A})$. It is enough to prove the result in $\mathcal{C}([0,T], \mathcal{M}_{0,A} \times \mathcal{M}_{\varepsilon,A} \times \mathcal{M}_{0,A})$ for all $\varepsilon > 0$ and $T > 0$. Let $\bar{\mu}^i = (\bar{\mu}^{\text{S},i}, \bar{\mu}^{\text{IS},i}, \bar{\mu}^{\text{RS},i})$, $i \in \{1,2\}$ be two solutions of (1.3)-(1.5), started with the same initial conditions. Set

$$\Upsilon_t = \sum_{j=0}^{3} |\langle \bar{\mu}_t^{\text{S},1}, \chi^j\rangle - \langle \bar{\mu}_t^{\text{S},2}, \chi^j\rangle| + \sum_{j=0}^{2}\Big(|\langle \bar{\mu}_t^{\text{IS},1}, \chi^j\rangle - \langle \bar{\mu}_t^{\text{IS},2}, \chi^j\rangle| + |\langle \bar{\mu}_t^{\text{RS},1}, \chi^j\rangle - \langle \bar{\mu}_t^{\text{RS},2}, \chi^j\rangle|\Big).$$

Let us first remark that for all $0 \le t < T$, $\bar{N}_t^{\text{S}} \ge \bar{N}_t^{\text{IS}} > \varepsilon$ and then

$$\begin{aligned} |\bar{p}_t^{\text{I},1} - \bar{p}_t^{\text{I},2}| = \Big|\frac{\bar{N}_t^{\text{IS},1}}{\bar{N}_t^{\text{S},1}} - \frac{\bar{N}_t^{\text{IS},2}}{\bar{N}_t^{\text{S},2}}\Big| &\le \frac{A}{\varepsilon^2}\Big|\bar{N}_t^{\text{S},1} - \bar{N}_t^{\text{S},2}\Big| + \frac{1}{\varepsilon}\Big|\bar{N}_t^{\text{IS},1} - \bar{N}_t^{\text{IS},2}\Big| \\ &= \frac{A}{\varepsilon^2}\Big|\langle \bar{\mu}_t^{\text{S},1}, \chi\rangle - \langle \bar{\mu}_t^{\text{S},2}, \chi\rangle\Big| + \frac{1}{\varepsilon}\Big|\langle \bar{\mu}_t^{\text{IS},1}, \chi\rangle - \langle \bar{\mu}_t^{\text{IS},2}, \chi\rangle\Big| \le \frac{A}{\varepsilon^2}\Upsilon_t. \quad (3.15) \end{aligned}$$

The same computations show a similar result for $|\bar{p}_t^{\text{S},1} - \bar{p}_t^{\text{S},2}|$.

Using that $\bar{\mu}^i$ are solutions to (1.3)-(1.4) let us show that Υ satisfies a Gronwall inequality which implies that it is equal to 0 for all $t \le T$. For the degree distributions of the susceptible individuals, we have for $p \in \{0,1,2,3\}$ and $f = \chi^p$ in (1.3):

$$\begin{aligned} |\langle \bar{\mu}_t^{\text{S},1}, \chi^p\rangle - \langle \bar{\mu}_t^{\text{S},2}, \chi^p\rangle| =& \Big|\sum_{k\in\mathbb{N}} \bar{\mu}_0^{\text{S}}(k) k^p \big(e^{-r\int_0^t \bar{p}_s^{\text{I},1} ds} - e^{-r\int_0^t \bar{p}_s^{\text{I},2} ds}\big)\Big| \\ \le& r\sum_{k\in\mathbb{N}} k^p \bar{\mu}_0^{\text{S}}(k)\int_0^t \big|\bar{p}_s^{\text{I},1} - \bar{p}_s^{\text{I},2}\big| ds \le r\frac{A^2}{\varepsilon^2}\int_0^t \Upsilon_s ds, \end{aligned}$$

by using (3.15) and the fact that $\bar{\mu}_0^{\text{S}} \in \mathcal{M}_{0,A}$.
For $\bar{\mu}^{\text{IS}}$ and $\bar{\mu}^{\text{RS}}$, we use (1.4) and (1.5) with the functions $f = \chi^0$, $f = \chi$ and $f = \chi^2$. We proceed here with only one of the computations, others can be done similarly. From (1.4):

$$\langle \bar{\mu}_t^{\text{IS},1}, \mathbf{1}\rangle - \langle \bar{\mu}_t^{\text{IS},2}, \mathbf{1}\rangle = \beta\int_0^t \langle \bar{\mu}_s^{\text{IS},1} - \bar{\mu}_s^{\text{IS},2}, \mathbf{1}\rangle \, \mathrm{d}s + r\int_0^t (\bar{p}_s^{\text{I},1}\langle \bar{\mu}_s^{\text{S},1}, \chi\rangle - \bar{p}_s^{\text{I},2}\langle \bar{\mu}_s^{\text{S},2}, \chi\rangle) \, \mathrm{d}s.$$

Hence, with (3.15),

$$\Big|\langle \bar{\mu}_t^{\text{IS},1} - \bar{\mu}_t^{\text{IS},2}, \mathbf{1}\rangle\Big| \le C(\beta, r, A, \varepsilon)\int_0^t \Upsilon_s \, \mathrm{d}s.$$

By analogous computations for the other quantities, we then show that

$$\Upsilon_t \le C'(\beta, r, A, \varepsilon) \int_0^t \Upsilon_s \, \mathrm{d}s,$$

hence $\Upsilon \equiv 0$. It follows that for all $t < T$, and for all $j \in \{0,1,2\}$,

$$\langle \bar{\mu}_t^{\mathrm{S},1}, \chi^j \rangle = \langle \bar{\mu}_t^{\mathrm{S},2}, \chi^j \rangle \quad \text{and} \quad \langle \bar{\mu}_t^{\mathrm{IS},1}, \chi^j \rangle = \langle \bar{\mu}_t^{\mathrm{IS},2}, \chi^j \rangle, \tag{3.16}$$

and in particular, $\bar{N}_t^{\mathrm{S},1} = \bar{N}_t^{\mathrm{S},2}$ and $\bar{N}_t^{\mathrm{IS},1} = \bar{N}_t^{\mathrm{IS},2}$. This implies that $\bar{p}_t^{\mathrm{S},1} = \bar{p}_t^{\mathrm{S},2}$, $\bar{p}_t^{\mathrm{I},1} = \bar{p}_t^{\mathrm{I},2}$ and $\bar{p}_t^{\mathrm{R},1} = \bar{p}_t^{\mathrm{R},2}$. From (1.3) and the continuity of the solutions to (1.3)-(1.5), pathwise uniqueness holds for $\bar{\mu}^{\mathrm{S}}$ a.s.

Our purpose is now to prove that $\bar{\mu}^{\mathrm{IS},1} = \bar{\mu}^{\mathrm{IS},2}$. Let us introduce the following generating functions: for any $t \in \mathbb{R}_+$, $i \in \{1,2\}$ and $\eta \in [0,1)$,

$$\mathcal{G}_t^i(\eta) = \sum_{k \ge 0} \eta^k \bar{\mu}_t^{\mathrm{IS},i}(k).$$

Since we already know these measures do have the same total mass, it boils down to prove that $\mathcal{G}^1 \equiv \mathcal{G}^2$. Let us define

$$\begin{aligned} H(t,\eta) &= \int_0^t \sum_{k\in\mathbb{N}} rk\bar{p}_s^{\mathrm{I}} \sum_{\substack{j,\ell,m\in\mathbb{N} \\ j+\ell+m=k-1}} \binom{k-1}{j,\ell,m} (\bar{p}_s^{\mathrm{I}})^j (\bar{p}_s^{\mathrm{R}})^\ell (\bar{p}_s^{\mathrm{S}})^m \eta^m \bar{\mu}_s^{\mathrm{S}}(k)\, ds \\ K_t &= \sum_{k\in\mathbb{N}} rk\bar{p}_t^{\mathrm{I}}(k-1)\bar{p}_t^{\mathrm{R}} \frac{\bar{\mu}_t^{\mathrm{S}}(k)}{\bar{N}_t^{\mathrm{IS}}}. \end{aligned} \tag{3.17}$$

The latter quantities are respectively of class $\mathcal{C}^1$ and $\mathcal{C}^0$ with respect to time t and are well-defined and bounded on $[0,T]$. Moreover, H and K do not depend on the chosen solution because of (3.16). Applying (1.4) to $f(k) = \eta^k$ yields

$$\begin{aligned} \mathcal{G}_t^i(\eta) =& \mathcal{G}_0^i(\eta) + H(t,\eta) + \int_0^t \Big(K_s \sum_{k'\in\mathbb{N}^*} \big(\eta^{k'-1} - \eta^{k'}\big) k' \bar{\mu}_s^{\mathrm{IS},i}(k') - \beta \mathcal{G}_s^i(\eta)\Big) ds \\ =& \mathcal{G}_0^i(\eta) + H(t,\eta) + \int_0^t \Big(K_s (1-\eta)\partial_\eta \mathcal{G}_s^i(\eta) - \beta \mathcal{G}_s^i(\eta)\Big) ds. \end{aligned}$$

Then, the functions $t \mapsto \widetilde{\mathcal{G}}_t^i(\eta)$ defined by $\widetilde{\mathcal{G}}_t^i(\eta) = e^{\beta t}\mathcal{G}_t^i(\eta)$, $i \in \{1,2\}$, are solutions of the following transport equation:

$$\partial_t g(t,\eta) - (1-\eta) K_t \, \partial_\eta g(t,\eta) = \partial_t H(t,\eta) e^{\beta t}. \tag{3.18}$$

In view of the regularity of H and K, it is known that this equation admits a unique solution (see *e.g.* [12]). Hence $\mathcal{G}_t^1(\eta) = \mathcal{G}_t^2(\eta)$ for all $t \in \mathbb{R}_+$ and $\eta \in [0,1)$. The same method applies to $\bar{\mu}^{\mathrm{RS}}$. Thus there is at most one solution to the differential system (1.3)-(1.5).

Step 3 We now show that $\mu^{(n)}$ nearly satisfies (1.3)-(1.5) as n gets large. Recall (3.5) for a bounded function f on $\mathbb{N}$. To identify the limiting values, we establish that for all $n \in \mathbb{N}^*$ and all $t \ge 0$,

$$\langle \mu_{t\wedge\tau_\varepsilon^n}^{(n),\mathrm{IS}}, f\rangle = \Psi_{t\wedge\tau_\varepsilon^n}^{\mathrm{IS},f}(\mu^{(n)}) + \Delta_{t\wedge\tau_\varepsilon^n}^{n,f} + M_{t\wedge\tau_\varepsilon^n}^{(n),\mathrm{IS},f}, \tag{3.19}$$

where $M^{(n),\mathrm{IS},f}$ is defined in (3.5) and where $\Delta_{.\wedge\tau_\varepsilon^n}^{n,f}$ converges to 0 when $n \to +\infty$, in probability and uniformly in t on compact time intervals.

Let us fix $t \in \mathbb{R}_+$. Computation similar to (3.13) give:

$$\mathbb{E}\big((M_t^{(n),\text{IS},f})^2\big) = \mathbb{E}\big(\langle M^{(n),\text{IS},f}\rangle_t\big) \leq \frac{(25r+2\beta)\, At\|f\|_\infty^2}{n}. \tag{3.20}$$

Hence the sequence $(M_t^{(n),\text{IS},f})_{n\in\mathbb{N}}$ converges in L^2 and in probability to zero (and in L^1 by Cauchy-Schwarz inequality).

We now consider the finite variation part of (3.5), given in (3.6). The sum in (3.6) corresponds to the links to I that the new infected individual has. We separate this sum into cases where the new infected individual only has simple edges to other individuals of I, and cases where multiple edges exist. The latter term is expected to vanish for large populations.

$$A_t^{(n),\text{IS},f} = B_t^{(n),\text{IS},f} + C_t^{(n),\text{IS},f}, \tag{3.21}$$

where

$$\begin{aligned} B_t^{(n),\text{IS},f} =& -\int_0^t \beta\langle \mu_s^{(n),\text{IS}}, f\rangle \,\mathrm{d}s \\ &+ \int_0^t \sum_{k\in\mathbb{N}} \lambda_s^n(k)\mu_s^{(n),\text{S}}(k) \sum_{j+\ell+1\leq k} p_s^n(j,\ell|k-1)\Bigg\{ f(k-(j+1+\ell)) \\ &+ \sum_{\substack{u\in\mathcal{U}(j+1,\mu_s^{n,\text{IS}});\\ \forall i\leq I_{s_-}^n,\, u_i\leq 1}} \rho(u|j+1,\mu_s^{n,\text{IS}}) \sum_{i=0}^{I_{s_-}^n} \Big(f\left(\gamma_i(\mu_{s_-}^{n,\text{IS}}) - u_i\right) - f\left(\gamma_i(\mu_{s_-}^{n,\text{IS}})\right)\Big) \Bigg\} \,\mathrm{d}s \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} C_t^{(n),\text{IS},f} =& \int_0^t \sum_{k\in\mathbb{N}} \lambda_s^n(k)\mu_s^{(n),\text{S}}(k) \sum_{j+\ell+1\leq k} p_s^n(j,\ell|k-1) \\ &\times \sum_{\substack{u\in\mathcal{U}(j+1,\mu_s^{n,\text{IS}});\\ \exists i\leq I_{s_-}^n,\, u_i> 1}} \rho(u|j+1,\mu_s^{n,\text{IS}}) \sum_{i=1}^{I_{s_-}^n} \Big(f\left(\gamma_i(\mu_{s_-}^{n,\text{IS}}) - u_i\right) - f\left(\gamma_i(\mu_{s_-}^{n,\text{IS}})\right)\Big) \,\mathrm{d}s. \end{aligned} \tag{3.23}$$

We first show that $C_t^{(n),\text{SI},f}$ is a negligible term. Let $q_{j,\ell,s}^n$ denote the probability that the newly infected individual at time s has a double (or of higher order) edge to some alter in $\text{I}_{s_-}^n$, given j and ℓ. The probability to have a multiple edge to a given infectious i is less than the number of couples of edges linking the newly infected to i, times the probability that these two particular edges linking i to a susceptible alter at s_- actually lead to the newly infected. Hence,

$$\begin{aligned} q_{j,\ell,s}^n =& \sum_{\substack{u\in\mathcal{U}(j+1,\mu_s^{n,\text{IS}});\\ \exists i\leq I_{s_-}^n,\, u_i> 1}} \rho(u|j+1,\mu_s^{n,\text{IS}}) \\ \leq& \binom{j}{2} \sum_{x\in\text{I}_{s_-}^n} \frac{d_x(\text{S}_{s_-}^n)(d_x(\text{S}_{s_-}^n)-1)}{N_{s_-}^{n,\text{IS}}(N_{s_-}^{n,\text{IS}}-1)} = \binom{j}{2}\frac{1}{n}\frac{\langle \mu_{s_-}^{(n),\text{IS}}, \chi(\chi-1)\rangle}{N_{s_-}^{(n),\text{IS}}(N_{s_-}^{(n),\text{IS}}-1/n)} \\ \leq& \binom{j}{2}\frac{1}{n}\frac{A}{\varepsilon(\varepsilon-1/n)} \qquad \text{if } s<\tau_\varepsilon^n \text{ and } n>1/\varepsilon. \end{aligned} \tag{3.24}$$

Then, since for all $u \in \mathcal{U}(j+1, \mu_s^{n,\text{IS}})$,

$$\Big| \sum_{i=1}^{I^n_{s_-}} \left(f\left(\gamma_i(\mu_{s_-}^{n,\text{IS}}) - u_i\right) - f\left(\gamma_i(\mu_{s_-}^{n,\text{IS}})\right)\right) \Big| \leq 2(j+1)\|f\|_\infty, \tag{3.25}$$

we have by (3.24) and (3.25), for $n > 1/\varepsilon$,

$$\begin{aligned} &|C^{(n),\text{IS},f}_{t\wedge\tau_\varepsilon^n}| \\ \leq &\int_0^{t\wedge\tau_\varepsilon^n} \sum_{k\in\mathbb{N}} rk\mu_s^{(n),\text{S}}(k) \sum_{j+\ell+1\leq k} p_s^n(j,\ell|k-1) 2(j+1)\|f\|_\infty \frac{j(j-1)A}{2n\varepsilon(\varepsilon-1/n)} \,\mathrm{d}s \\ \leq &\frac{A\,rt\|f\|_\infty}{n\,\varepsilon(\varepsilon-1/n)} \langle \mu_0^{(n),\text{S}}, \chi^4\rangle, \end{aligned} \tag{3.26}$$

which tends to zero in view of (3.10) and thanks to the fact that $\mu_s^{(n),\text{S}}$ is dominated by $\mu_0^{(n),\text{S}}$ for all $s \geq 0$ and $n \in \mathbb{N}^*$.

We now aim at proving that $B^{(n),\text{IS},f}_{.\wedge\tau_\varepsilon^n}$ is somewhat close to $\Psi^{\text{IS},f}_{.\wedge\tau_\varepsilon^n}(\mu^{(n)})$. First, notice that

$$\begin{aligned} &\sum_{\substack{u\in\mathcal{U}(j+1,\mu_s^{n,\text{IS}});\\ \forall i\leq I^n_{s_-},\, u_i\leq 1}} \rho(u|j+1,\mu_s^{n,\text{IS}}) \sum_{i=1}^{I^n_{s_-}} \left(f\left(\gamma_i(\mu_{s_-}^{n,\text{IS}}) - u_i\right) - f\left(\gamma_i(\mu_{s_-}^{n,\text{IS}})\right)\right) \\ &\qquad = \sum_{\substack{u\in(\text{I}^n_{s_-})^{j+1}\\ u_0\neq\cdots\neq u_j}} \left(\frac{\prod_{k=0}^j d_{u_k}(\text{S}^n_s)}{N^{n,\text{IS}}_{s_-}\dots(N^{n,\text{SI}}_{s_-}-(j+1))}\right) \\ &\qquad\qquad \times \sum_{m=0}^j \left(f\left(d_{u_m}(\text{S}^n_{s_-})-1\right) - f\left(d_{u_m}(\text{S}^n_{s_-})\right)\right) \\ &\qquad = \sum_{m=0}^j \sum_{\substack{u\in(\text{I}^n_{s_-})^{j+1}\\ u_0\neq\cdots\neq u_j}} \left(\frac{\prod_{k=0}^j d_{u_k}(\text{S}^n_s)}{N^{n,\text{IS}}_{s_-}\dots(N^{n,\text{SI}}_{s_-}-(j+1))}\right) \\ &\qquad\qquad \times \left(f\left(d_{u_m}(\text{S}^n_{s_-})-1\right) - f\left(d_{u_m}(\text{S}^n_{s_-})\right)\right) \\ &\qquad = \sum_{m=0}^j \left(\sum_{x\in\text{I}^n_{s_-}} \frac{d_x(\text{S}^n_{s_-})}{N^{n,\text{IS}}_{s_-}} \left(f\left(d_x(\text{S}^n_{s_-})-1\right) - f\left(d_x(\text{S}^n_{s_-})\right)\right)\right) \\ &\qquad\qquad \times \left(\sum_{\substack{u\in\left(\text{I}^n_{s_-}\setminus\{x\}\right)^j\\ u_0\neq\cdots\neq u_{j-1}}} \frac{\prod_{k=0}^{j-1} d_{u_k}(\text{S}^n_s)}{(N^{n,\text{IS}}_{s_-}-1)\dots(N^{n,\text{IS}}_{s_-}-(j+1))}\right) \\ &\qquad = (j+1)\frac{\langle \mu_{s_-}^{(n),\text{IS}}, \chi\,(\tau_1 f - f)\rangle}{N^{(n),\text{IS}}_{s_-}} \left(1 - q^n_{j-1,\ell,s}\right), \end{aligned} \tag{3.27}$$

where we recall that $\tau_1 f(k) = f(k-1)$ for every function f on $\mathbb{N}$ and $k \in \mathbb{N}$. In the third equality, we split the term u_m from the other terms $(u_{m'})_{m'\neq m}$. The last sum in the r.h.s. of this equality is the probability of drawing j different infectious individuals that are not u_m and that are all different, hence $1 - q^n_{j-1,\ell,s}$.

Denote for $t > 0$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
p_t^{n,\mathrm{I}} &= \frac{\langle \mu_t^{n,\mathrm{IS}}, \chi\rangle - 1}{\langle \mu_t^{n,\mathrm{S}}, \chi\rangle - 1},\\
p_t^{n,\mathrm{R}} &= \frac{\langle \mu_t^{n,\mathrm{RS}}, \chi\rangle}{\langle \mu_t^{n,\mathrm{S}}, \chi\rangle - 1},\\
p_t^{n,\mathrm{S}} &= \frac{\langle \mu_t^{n,\mathrm{S}}, \chi\rangle - \langle \mu_t^{n,\mathrm{IS}}, \chi\rangle - \langle \mu_t^{n,\mathrm{RS}}, \chi\rangle}{\langle \mu_t^{n,\mathrm{S}}, \chi\rangle - 1},
\end{aligned}
$$

the proportion of edges with infectious (resp. removed and susceptible) alters and susceptible egos among all the edges with susceptible egos but the contaminating edge. For all integers j and ℓ such that $j + \ell \leq k - 1$ and $n \in \mathbb{N}^*$, denote by

$$
\tilde{p}_t^n(j, \ell \mid k-1) = \frac{(k-1)!}{j!(k-1-j-\ell)!\ell!}(p_t^{n,\mathrm{I}})^j (p_t^{n,\mathrm{R}})^\ell (p_t^{n,\mathrm{S}})^{k-1-j-\ell},
$$

the probability that the multinomial variable counting the number of edges with infectious, removed and susceptible alters, among $k-1$ given edges, equals $(j, \ell, k-1-j-\ell)$. We have that

$$
|\Psi^{\mathrm{IS},f}_{t\wedge\tau_\varepsilon^n}(\mu^{(n)}) - B^{(n),\mathrm{IS},f}_{t\wedge\tau_\varepsilon^n}| \leq |D^{(n),\mathrm{IS},f}_{t\wedge\tau_\varepsilon^n}| + |E^{(n),\mathrm{IS},f}_{t\wedge\tau_\varepsilon^n}|, \tag{3.28}
$$

where

$$
\begin{aligned}
D_t^{(n),\mathrm{IS},f} =& \int_0^t \sum_{k\in\mathbb{N}} \lambda_s^n(k)\mu_s^{(n),\mathrm{S}}(k) \sum_{j+\ell+1\leq k} (p_s^n(j,\ell|k-1) - \tilde{p}_s^n(j,\ell|k-1))\\
&\times \left(f(k-(j+\ell+1)) + (j+1)\frac{\langle \mu_{s_-}^{(n),\mathrm{IS}}, \chi\big(\tau_1 f - f\big)\rangle}{N_{s_-}^{(n),\mathrm{IS}}}\right) \mathrm{d}s,\\
E_t^{(n),\mathrm{IS},f} =& \int_0^t \sum_{k\in\mathbb{N}} \lambda_s^n(k)\mu_s^{(n),\mathrm{S}}(k)\\
&\times \sum_{j+\ell+1\leq k} p_s^n(j,\ell|k-1)(j+1)\frac{\langle \mu_{s_-}^{(n),\mathrm{IS}}, \chi\big(\tau_1 f - f\big)\rangle}{N_{s_-}^{(n),\mathrm{IS}}} q^n_{j-1,\ell,s} \ \mathrm{d}s.
\end{aligned}
$$

First,

$$
|D^{(n),\mathrm{IS},f}_{t\wedge\tau_\varepsilon^n}| \leq \int_0^{t\wedge\tau_\varepsilon^n} \sum_{k\in\mathbb{N}} rk\alpha_s^n(k)\|f\|_\infty \left(1 + \frac{2kA}{\varepsilon}\right) \mu_s^{(n),\mathrm{S}}(k) \ \mathrm{d}s, \tag{3.29}
$$

where for all $k \in \mathbb{N}$

$$
\alpha_t^n(k) = \sum_{j+\ell+1\leq k} \left| p_t^n(j,\ell|k-1) - \tilde{p}_t^n(j,\ell|k-1) \right|.
$$

The multinomial probability $\tilde{p}_s^n(j,\ell|k-1)$ approximates the hypergeometric one, $p_s^n(j,\ell|k-1,s)$, as n increases to infinity, in view of the fact that the total population size, $\langle \mu_0^{n,\mathrm{S}}, \mathbf{1}\rangle + \langle \mu_0^{n,\mathrm{IS}}, \mathbf{1}\rangle$, is of order n. Hence, the r.h.s. of (3.29) vanishes by dominated convergence.

On another hand, using (3.24),

$$
\begin{aligned}
|E^{(n),\mathrm{IS},f}_{t\wedge\tau_\varepsilon^n}| \leq& \int_0^{t\wedge\tau_\varepsilon^n} \sum_{k\in\mathbb{N}} rk^2\mu_s^{(n),\mathrm{S}}(k)\frac{2\|f\|_\infty A}{\varepsilon}\frac{k^2 A}{2n\varepsilon(\varepsilon - 1/n)} \ \mathrm{d}s\\
\leq& \frac{A^3 rt\|f\|_\infty}{n\varepsilon^2(\varepsilon - 1/n)},
\end{aligned} \tag{3.30}
$$

in view of (3.10). Gathering (3.20), (3.21), (3.26), (3.28), (3.29) and (3.30) concludes the proof that the rest of (3.19) vanishes in probability uniformly over compact intervals.

Step 4 Recall that in this proof, $\bar{\mu} = (\bar{\mu}^{\mathrm{S}}, \bar{\mu}^{\mathrm{IS}}, \bar{\mu}^{\mathrm{RS}})$ is the limit of $\mu^{(n)}_{.\wedge\tau^n_\varepsilon} = (\mu^{(n),\mathrm{S}}_{.\wedge\tau^n_\varepsilon}, \mu^{(n),\mathrm{IS}}_{.\wedge\tau^n_\varepsilon}, \mu^{(n),\mathrm{RS}}_{.\wedge\tau^n_\varepsilon})_{n\in\mathbb{N}^*}$, and recall that these processes take values in the closed set $\mathcal{M}^3_{0,A}$. Our purpose is now to prove that $\bar{\mu}$ satisfy (1.3)-(1.5). Using Skorokhod representation theorem, there exists, on the same probability space as $\bar{\mu}$, a sequence, again denoted by $(\mu^{(n)}_{.\wedge\tau^n_\varepsilon})_{n\in\mathbb{N}^*}$ with an abuse of notation, with same marginal distributions as the original sequence, and that converges a.s. to $\bar{\mu}$.

The maps $\nu_. := (\nu^1_., \nu^2_., \nu^3_.) \mapsto \langle \nu^1_., \mathbf{1}\rangle/(\langle \nu^1_0, \mathbf{1}\rangle + \langle \nu^2_0, \mathbf{1}\rangle + \langle \nu^3_0, \mathbf{1}\rangle)$ (respectively $\langle \nu^2_., \mathbf{1}\rangle/(\langle \nu^1_0, \mathbf{1}\rangle + \langle \nu^2_0, \mathbf{1}\rangle + \langle \nu^3_0, \mathbf{1}\rangle)$ and $\langle \nu^3_., \mathbf{1}\rangle/(\langle \nu^1_0, \mathbf{1}\rangle + \langle \nu^2_0, \mathbf{1}\rangle + \langle \nu^3_0, \mathbf{1}\rangle)$) are continuous from $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_{0,A}\times\mathcal{M}_{\varepsilon,A}\times\mathcal{M}_{0,A})$ into $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$.
Then, Lemma A.5 together with the continuity of $(X^1_., X^2_.) \mapsto X^1_./X^2_.$ from $\mathcal{C}(\mathbb{R}_+, \mathbb{R})\times\mathcal{C}(\mathbb{R}_+, \mathbb{R}^*)$ into $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ (see *e.g.* [28]) implies that the mapping $\nu_. \mapsto \langle \nu^1_., \chi\rangle/\langle \nu^2_., \chi\rangle$ is continuous from $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_{0,A}\times\mathcal{M}_{\varepsilon,A}\times\mathcal{M}_{0,A})$ into $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$. The same argument yields the continuity of $\nu_. \mapsto \mathbb{1}_{\langle \nu^1_., \chi\rangle > \varepsilon}/\langle \nu^2_., \chi\rangle$ for the same spaces.
Lemma A.5 also provides the continuity of $\nu_. \mapsto \langle \nu^2_., \chi(\tau_1 f - f)\rangle$ from $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_{0,A}\times\mathcal{M}_{\varepsilon,A}\times\mathcal{M}_{0,A})$ into $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ for bounded function f on $\mathbb{N}$.
Since, as well known, the mapping $y \in \mathcal{D}([0,t], \mathbb{R}) \mapsto \int_0^t y_s \,\mathrm{d}s$ is continuous, we have proven the continuity of the mapping Ψ^f_t defined in (3.14) on $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{0,A}\times\mathcal{M}_{\varepsilon,A}\times\mathcal{M}_{0,A})$.

By Lemma A.5 applied to $\varphi = \chi$, the process $(N^{(n),\mathrm{IS}}_{.\wedge\tau^n_\varepsilon})_{n\in\mathbb{N}^*}$ converges in distribution to $\bar{N}^{\mathrm{IS}}_. = \langle \bar{\mu}^{\mathrm{IS}}_., \chi\rangle$. Since the latter process is continuous, the convergence holds in $(\mathcal{D}([0,T], \mathbb{R}_+), \|.\|_\infty)$ for any $T > 0$ (see [6] p. 112). As $y \in \mathcal{D}(\mathbb{R}_+, \mathbb{R}) \mapsto \inf_{t\in[0,T]} y(t) \in \mathbb{R}$ is continuous, we have a.s. that:

$$\inf_{t\in[0,T]} \bar{N}^{\mathrm{IS}}_t = \lim_{n\to+\infty} \inf_{t\in[0,T]} N^{(n),\mathrm{IS}}_{t\wedge\tau^n_\varepsilon} \quad (\geq \varepsilon).$$

We consider $\bar{t}_{\varepsilon'} = \inf\{t \in \mathbb{R}_+,\ \bar{N}^{\mathrm{IS}}_t \leq \varepsilon'\}$. A difficulty lies in the fact that we do not know yet whether this time is deterministic. We have a.s.:

$$\varepsilon' \leq \inf_{t\in[0,T]} \bar{N}^{\mathrm{IS}}_{t\wedge\bar{t}_{\varepsilon'}} = \lim_{n\to+\infty} \inf_{t\in[0,T]} N^{(n),\mathrm{IS}}_{t\wedge\tau^n_\varepsilon\wedge\bar{t}_{\varepsilon'}}. \tag{3.31}$$

Hence, using Fatou's lemma:

$$\begin{aligned} 1 =& \mathbb{P}\Big(\inf_{t\in[0,\bar{t}_{\varepsilon'}]} \bar{N}^{\mathrm{IS}}_t > \varepsilon\Big) \\ \leq& \lim_{n\to+\infty} \mathbb{P}\Big(\inf_{t\in[0,T\wedge\bar{t}_{\varepsilon'}]} N^{(n),\mathrm{IS}}_{t\wedge\tau^n_\varepsilon} > \varepsilon\Big) = \lim_{n\to+\infty} \mathbb{P}\Big(\tau^n_\varepsilon > T\wedge\bar{t}_{\varepsilon'}\Big). \end{aligned} \tag{3.32}$$

We have hence

$$\Psi^{\mathrm{IS},f}_{.\wedge\tau^n_\varepsilon\wedge\bar{t}_{\varepsilon'}\wedge T}(\mu^{(n)}) = \Psi^{\mathrm{IS},f}_{.\wedge\tau^n_\varepsilon\wedge T}(\mu^{(n)})\mathbb{1}_{\tau^n_\varepsilon\leq\bar{t}_{\varepsilon'}\wedge T} + \Psi^{\mathrm{IS},f}_{.\wedge\bar{t}_{\varepsilon'}\wedge T}(\mu^{(n)}_{.\wedge\tau^n_\varepsilon})\mathbb{1}_{\tau^n_\varepsilon>\bar{t}_{\varepsilon'}\wedge T}.$$

From the estimates of the different terms in (3.19), $\Psi^{\mathrm{IS},f}_{.\wedge\tau^n_\varepsilon\wedge T}(\mu^{(n)})$ is upper bounded by a moment of $\mu^{(n)}$ of order 4. In view of (3.10) and (3.32), the first term in the r.h.s. converges in L^1 and hence in probability to zero. Using the continuity of $\Psi^{\mathrm{IS},f}$ on $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{0,A}\times\mathcal{M}_{\varepsilon,A}\times\mathcal{M}_{0,A})$, $\Psi^{\mathrm{IS},f}(\mu^{(n)}_{.\wedge\tau^n_\varepsilon})$ converges to $\Psi^{\mathrm{IS},f}(\bar{\mu})$ and therefore, $\Psi^{\mathrm{IS},f}_{.\wedge\bar{t}_{\varepsilon'}\wedge T}(\mu^{(n)}_{.\wedge\tau^n_\varepsilon})$ converges to $\Psi^{\mathrm{IS},f}_{.\wedge\bar{t}_{\varepsilon'}\wedge T}(\bar{\mu})$. Thanks to this and (3.32), the second term in the r.h.s. converges to $\Psi^{\mathrm{IS},f}_{.\wedge\bar{t}_{\varepsilon'}\wedge T}(\bar{\mu})$ in $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$.
Then, $(\langle \mu^{(n),\mathrm{IS}}_{.\wedge\tau^n_\varepsilon\wedge\bar{t}_{\varepsilon'}\wedge T}, f\rangle - \Psi^{\mathrm{IS},f}_{.\wedge\tau^n_\varepsilon\wedge\bar{t}_{\varepsilon'}\wedge T}(\mu^{(n)}))_{n\in\mathbb{N}^*}$ converges in probability to $\langle \bar{\mu}^{\mathrm{IS}}_{.\wedge\bar{t}_{\varepsilon'}\wedge T}, f\rangle - \Psi^{\mathrm{IS},f}_{.\wedge\bar{t}_{\varepsilon'}\wedge T}(\bar{\mu})$.
From (3.19), this sequence also converges in probability to zero.

By identification of these limits, $\bar{\mu}^{\text{IS}}$ solves (1.4) on $[0, \bar{t}_{\varepsilon'} \wedge T]$. If $\langle \bar{\mu}_0^{\text{RS}}, \chi \rangle > 0$ then similar techniques can be used. Else, the result is obvious since for all $t \in [0, t_{\varepsilon'} \wedge T]$, $\langle \mu_t^{(n),\text{IS}}, \chi \rangle > \varepsilon$ and the term $p_t^n(j, \ell | k-1)$ is negligible when $\ell > 0$. Thus $\bar{\mu}$ coincides a.s. with the only continuous deterministic solution of (1.3)-(1.5) on $[0, \bar{t}_{\varepsilon'} \wedge T]$. This implies that $\bar{t}_{\varepsilon'} \wedge T = t_{\varepsilon'} \wedge T$ and yields the convergence in probability of $(\mu^{(n)}_{.\wedge \tau_\varepsilon^n})_{n \in \mathbb{N}^*}$ to $\bar{\mu}$, uniformly on $[0, t_{\varepsilon'} \wedge T]$ since $\bar{\mu}$ is continuous.

We finally prove that the non-localized sequence $(\mu^{(n)})_{n \in \mathbb{N}^*}$ also converges uniformly and in probability to $\bar{\mu}$ in $\mathcal{D}\left([0, t_{\varepsilon'}], \mathcal{M}_{0,A} \times \mathcal{M}_{\varepsilon,A} \times \mathcal{M}_{0,A}\right)$. For a small positive η,

$$\begin{aligned}
\mathbb{P}\Big(\sup_{t \in [0, t_{\varepsilon'}]} \Big| \langle \mu_t^{(n),\text{IS}}, f \rangle - \Psi_t^{\text{IS},f}(\bar{\mu}) \Big| > \eta \Big) \\
\leq \mathbb{P}\Big(\sup_{t \in [0, t_{\varepsilon'}]} \Big| \Psi_{t \wedge \tau_\varepsilon^n}^{\text{IS},f}(\mu^{(n)}) - \Psi_t^{\text{IS},f}(\bar{\mu}) \Big| > \frac{\eta}{2} \, ; \, \tau_\varepsilon^n \geq t_{\varepsilon'} \Big) \\
+ \mathbb{P}\Big(\sup_{t \in [0, t_{\varepsilon'}]} \Big| \Delta_{t \wedge \tau_\varepsilon^n}^{n,f} + M_{t \wedge \tau_\varepsilon^n}^{(n),\text{IS},f} \Big| > \frac{\eta}{2} \Big) + \mathbb{P}\Big(\tau_\varepsilon^n < t_{\varepsilon'} \Big). \quad (3.33)
\end{aligned}$$

Using the continuity of Ψ^f and the uniform convergence in probability proved above, the first term in the r.h.s. of (3.33) converges to zero. We can show that the second term converges to zero by using Doob's inequality together with the estimates of the bracket of $M^{(n),\text{IS},f}$ (similar to (3.13)) and of $\Delta^{n,f}$ (Step 2). Finally, the third term vanishes in view of (3.32).

The convergence of the original sequence $(\mu^{(n)})_{n \in \mathbb{N}^*}$ is then entailed by the uniqueness of the solution to (1.3)-(1.5), implied by Step 2.

Step 5 When $n \to +\infty$, by taking the limit in (3.1), $\left(\mu^{(n),\text{S}}\right)_{n \in \mathbb{N}^*}$ converges in $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{0,A})$ to the solution of the following transport equation, that can be solved in function of $\bar{p}^{\text{I}}$. For every bounded function $f : (k,t) \mapsto f_t(k) \in \mathcal{C}_b^{0,1}(\mathbb{N} \times \mathbb{R}_+, \mathbb{R})$ of class $\mathcal{C}^1$ with bounded derivative with respect to t,

$$\langle \bar{\mu}_t^{\text{S}}, f_t \rangle = \langle \bar{\mu}_0^{\text{S}}, f_0 \rangle - \int_0^t \langle \bar{\mu}_s^{\text{S}}, r\chi \bar{p}_s^{\text{I}} f_s - \partial_s f_s \rangle \, \mathrm{d}s. \quad (3.34)$$

Choosing $f(k,s) = \varphi(k) \exp\left(-rk \int_0^{t-s} \bar{p}^{\text{I}}(u) du \right)$, we obtain that

$$\langle \bar{\mu}_t^{\text{S}}, \varphi \rangle = \sum_{k \in \mathbb{N}} \varphi(k) \theta_t^k \bar{\mu}_0^{\text{S}}(k). \quad (3.35)$$

where $\theta_t = \exp\left(-r \int_0^t \bar{p}^{\text{I}}(u) du \right)$ is the probability that a given degree 1 node remains susceptible at time t. This is the announced Equation (1.3). ■

We end this section with a lower bound of the time $t_{\varepsilon'}$ until which we proved that the convergence to Volz' equations holds.

Proposition 3.3. *Under the assumptions of Theorem 1,*

$$t_{\varepsilon'} > \bar{\tau}_{\varepsilon'} := \frac{\log\left(\langle \bar{\mu}_0^{\text{S}}, \chi^2 \rangle + \bar{N}_0^{\text{IS}} \right) - \log\left(\langle \bar{\mu}_0^{\text{S}}, \chi^2 \rangle + \varepsilon' \right)}{\max(\beta, r)}. \quad (3.36)$$

Proof. Because of the moment Assumption (3.10), we can prove that (3.19) also holds for $f = \chi$. This is obtained by replacing in (3.20), (3.26), (3.29) and (3.30) $\|f\|_\infty$ by k and using the Assumption of boundedness of the moments of order 5 in (3.26) and (3.30). This shows that $(N^{(n),\text{IS}})_{n \in \mathbb{N}}$ converges, uniformly on $[0, t_{\varepsilon'}]$ and in probability, to the deterministic and continuous solution $\bar{N}^{\text{IS}} = \langle \bar{\mu}^{\text{IS}}, \chi \rangle$. We introduce the event $\mathcal{A}^n_\xi = \{|\ N^{n,\text{IS}}_0 - n\bar{N}^{\text{IS}}_0\ | \leq \xi\}$ where their differences are bounded by $\xi > 0$. Recall the definition (3.9) and let us introduce the number of edges Z^n_t that were IS at time 0 and that have been removed before t. For $t \geq \tau^n_{\varepsilon'}$, we have necessarily that $Z^n_t \geq N^{n,\text{IS}}_0 - n\varepsilon'$. Thus,

$$\begin{aligned}\mathbb{P}\left(\{\tau^n_{\varepsilon'} \leq t\} \cap \mathcal{A}^n_\xi\right) \leq & \mathbb{P}\left(\{Z^n_t > N^{n,\text{IS}}_0 - n\varepsilon'\} \cap \mathcal{A}^n_\xi\right) \\ \leq & \mathbb{P}\left(\{Z^n_t > n(\bar{N}^{\text{IS}}_0 - \varepsilon') - \xi\} \cap \mathcal{A}^n_\xi\right). \end{aligned} \tag{3.37}$$

When susceptible (resp. infectious) individuals of degree k are contaminated (resp. removed), at most k IS-edges are lost. Let $X^{n,k}_t$ be the number of edges that, at time 0, are IS with susceptible alter of degree k, and that have transmitted the disease before time t. Let $Y^{n,k}_t$ be the number of initially infectious individuals x with $d_x(\text{S}_0) = k$ and who have been removed before time t. $X^{n,k}_t$ and $Y^{n,k}_t$ are bounded by $k\mu^{n,\text{S}}_0(k)$ and $\mu^{n,\text{IS}}_0(k)$. Thus:

$$Z^n_t \leq \sum_{k \in \mathbb{N}} k\big(X^{n,k}_t + Y^{n,k}_t\big). \tag{3.38}$$

Let us stochastically upper bound Z^n_t. Since each IS-edge transmits the disease independently at rate r, $X^{n,k}_t$ is stochastically dominated by a binomial r.v. of parameters $k\mu^{n,\text{S}}_0(k)$ and $1 - e^{-rt}$. We proceed similarly for $Y^{n,k}_t$. Conditionally to the initial condition, $X^{n,k}_t + Y^{n,k}_t$ is thus stochastically dominated by a binomial r.v. $\tilde{Z}^{n,k}_t$ of parameters $(k\mu^{n,\text{S}}_0(k) + \mu^{n,\text{IS}}_0(k))$ and $1 - e^{-\max(\beta,r)t}$. Then (3.37) and (3.38) give:

$$\mathbb{P}\left(\{\tau^n_{\varepsilon'} \leq t\} \cap \mathcal{A}^n_\xi\right) \leq \mathbb{P}\left(\sum_{k \in \mathbb{N}} \frac{k\tilde{Z}^{n,k}_t}{n} > \bar{N}^{\text{IS}}_0 - \varepsilon' - \frac{\xi}{n}\right). \tag{3.39}$$

Thanks to Assumption 3.1 and (3.10), the series $\sum_{k\in\mathbb{N}} k\tilde{Z}^{n,k}_t/n$ converges in L^1 and hence in probability to $(\langle \bar{\mu}^{\text{S}}_0, \chi^2 \rangle + \bar{N}^{\text{IS}}_0)(1 - e^{-\max(\beta,r)t})$ when $n \to +\infty$. Thus, for sufficiently large n,

$$\mathbb{P}\left(\{\tau^n_{\varepsilon'} \leq t\} \cap \mathcal{A}^n_\xi\right) = 1 \text{ if } t > \bar{\tau}_{\varepsilon'} \text{ and } 0 \text{ if } t < \bar{\tau}_{\varepsilon'}.$$

For all $t < \bar{\tau}_{\varepsilon'}$, it follows from Assumption 3.1, (3.10) and Lemma A.4 that:

$$\lim_{n\to+\infty} \mathbb{P}\left(\tau^n_{\varepsilon'} \leq t\right) \leq \lim_{n\to+\infty} \Big(\mathbb{P}\left(\{\tau^n_{\varepsilon'} \leq t\} \cap \mathcal{A}^n_\xi\right) + \mathbb{P}\big((\mathcal{A}^n_\xi)^c\big)\Big) = 0,$$

so that by Theorem 1

$$1 = \lim_{n\to+\infty} \mathbb{P}\left(\tau^n_{\varepsilon'} \geq \bar{\tau}_{\varepsilon'}\right) = \lim_{n\to+\infty} \mathbb{P}\Big(\inf_{t \leq \bar{\tau}_{\varepsilon'}} N^{(n),\text{IS}}_t \geq \varepsilon'\Big) = \mathbb{P}\Big(\inf_{t \leq \bar{\tau}_{\varepsilon'}} \bar{N}^{\text{IS}}_t \geq \varepsilon'\Big).$$

This shows that $t_{\varepsilon'} \geq \bar{\tau}_{\varepsilon'}$ a.s., which concludes the proof. ■

3.3 Proof of Volz' equations

Proposition 3.4. *The system* (1.3)-(1.5) *implies Volz' equations* (1.8)-(1.11).

Before proving Proposition 3.4, we begin with a corollary of Theorem 1.

Corollary 3.5. *For all* $t \in \mathbb{R}_+$

$$\begin{aligned}\bar{N}^{\text{S}}_t =& \theta_t g'(\theta_t) \\ \bar{N}^{\text{IS}}_t =& \bar{N}^{\text{IS}}_0 + \int_0^t r\bar{p}^{\text{I}}_s \theta_s g'(\theta_s)\Big((\bar{p}^{\text{S}}_s - \bar{p}^{\text{I}}_s)\theta_s \frac{g''(\theta_s)}{g'(\theta_s)} - 1\Big) - \beta\bar{N}^{\text{IS}}_s \ ds \\ \bar{N}^{\text{RS}}_t =& \int_0^t \Big(\beta\bar{N}^{\text{IS}}_s - r\bar{p}^{\text{R}}_s\bar{p}^{\text{I}}_s\theta_s^2 g''(\theta_s)\Big)\ ds. \end{aligned} \tag{3.40}$$

Proof. In the proof of Proposition 3.3, we have shown that $(N^{(n),\mathrm{IS}})_{n\in\mathbb{N}}$ converges uniformly on compact intervals and in probability to the deterministic and continuous solution $\bar{N}^{\mathrm{IS}} = \langle \bar{\mu}^{\mathrm{IS}}, \chi\rangle$. (1.3) with $f=\chi$ reads

$$\bar{N}^{\mathrm{S}}_t = \sum_{k\in\mathbb{N}} \bar{\mu}^{\mathrm{S}}_0(k) k \theta_t^k = \theta_t \sum_{k=1}^{+\infty} \bar{\mu}^{\mathrm{S}}_0(k) k\theta_t^{k-1} = \theta_t g'(\theta_t), \tag{3.41}$$

i.e. the first assertion of (3.40).

Choosing $f=\chi$ in (1.4), we obtain

$$\bar{N}^{\mathrm{IS}}_t = \bar{N}^{\mathrm{IS}}_0 - \int_0^t \beta \bar{N}^{\mathrm{IS}}_s \,\mathrm{d}s + \int_0^t \sum_{k\in\mathbb{N}} \lambda_s(k) \sum_{j+\ell\le k-1} \big(k-2j-2-\ell\big) \\ \times \Big[\frac{(k-1)!}{j!(k-1-j-\ell)!\ell!} (\bar{p}^{\mathrm{I}}_s)^j (\bar{p}^{\mathrm{R}}_s)^\ell (\bar{p}^{\mathrm{S}}_s)^{k-1-j-\ell}\Big] \bar{\mu}^{\mathrm{S}}_s(k)\,\mathrm{d}s.$$

Notice that the term in the square brackets is the probability to obtain $(j,\ell,k-1-j-\ell)$ from a draw in the multinomial distribution of parameters $(k-1,(\bar{p}^{\mathrm{I}}_s,\bar{p}^{\mathrm{R}}_s,\bar{p}^{\mathrm{S}}_s))$. Hence,

$$\sum_{j+\ell\le k-1} j\times \Big(\frac{(k-1)!}{j!(k-1-j-\ell)!\ell!} (\bar{p}^{\mathrm{I}}_s)^j (\bar{p}^{\mathrm{R}}_s)^\ell (\bar{p}^{\mathrm{S}}_s)^{k-1-j-\ell}\Big) = (k-1)\bar{p}^{\mathrm{I}}_s$$

as we recognize the mean number of edges to I_s of an individual of degree k. Other terms are treated similarly. Hence, with the definition of $\lambda_s(k)$, (2.1),

$$\bar{N}^{\mathrm{IS}}_t = \bar{N}^{\mathrm{IS}}_0 + \int_0^t r\,\bar{p}^{\mathrm{I}}_s \Big(\langle \bar{\mu}^{\mathrm{S}}_s, \chi^2-2\chi\rangle - (2\bar{p}^{\mathrm{I}}_s + \bar{p}^{\mathrm{R}}_s)\langle \bar{\mu}^{\mathrm{S}}_s, \chi(\chi-1)\rangle\Big)\,\mathrm{d}s \\ - \int_0^t \beta \bar{N}^{\mathrm{IS}}_s\,\mathrm{d}s.$$

But since

$$\langle \bar{\mu}^{\mathrm{S}}_t, \chi(\chi-1)\rangle = \sum_{k\in\mathbb{N}} \bar{\mu}^{\mathrm{S}}_0(k) k(k-1)\theta_t^k = \theta_t^2 g''(\theta_t) \\ \langle \bar{\mu}^{\mathrm{S}}_t, \chi^2-2\chi\rangle = \langle \bar{\mu}^{\mathrm{S}}_t, \chi(\chi-1)\rangle - \langle \bar{\mu}^{S}_t, \chi\rangle = \theta_t^2 g''(\theta_t) - \theta_t g'(\theta_t),$$

we obtain by noticing that $1-2\bar{p}^{\mathrm{I}}_s - \bar{p}^{\mathrm{R}}_s = \bar{p}^{\mathrm{S}}_s - \bar{p}^{\mathrm{I}}_s$,

$$\bar{N}^{\mathrm{IS}}_t = \bar{N}^{\mathrm{IS}}_0 + \int_0^t r\,\bar{p}^{\mathrm{I}}_s \Big((\bar{p}^{\mathrm{S}}_s - \bar{p}^{\mathrm{I}}_s)\theta_s^2 g''(\theta_s) - \theta_s g'(\theta_s)\Big)\,\mathrm{d}s - \int_0^t \beta \bar{N}^{\mathrm{IS}}_s\,\mathrm{d}s \tag{3.42}$$

which is the second assertion of (3.40). The third equation of (3.40) is obtained similarly. ∎

We are now ready to prove Volz' equations:

Proof of Proposition 3.4. We begin with the proof of (1.8) and (1.9). Fix again $t\ge 0$. For the size of the susceptible population, taking $\varphi = \mathbf{1}$ in (1.3), we are lead to introduce the same quantity $\theta_t = \exp(-r\int_0^t \bar{p}^{\mathrm{I}}_s ds)$ as Volz and obtain (1.8). For the size of the infective population, setting $f=\mathbf{1}$ in (1.4) entails

$$\bar{I}_t = \bar{I}_0 + \int_0^t \Big(\sum_{k\in\mathbb{N}} rk\bar{p}^{\mathrm{I}}_s \bar{\mu}^{\mathrm{S}}_s(k) - \beta \bar{I}_s\Big)\,\mathrm{d}s \\ = \bar{I}_0 + \int_0^t \Big(r\bar{p}^{\mathrm{I}}_s \sum_{k\in\mathbb{N}} \bar{\mu}^{\mathrm{S}}_0(k) k\theta_s^k - \beta\bar{I}_s\Big)\,\mathrm{d}s = \bar{I}_0 + \int_0^t \Big(r\bar{p}^{\mathrm{I}}_s \theta_s g'(\theta_s) - \beta \bar{I}_s\Big)\,\mathrm{d}s$$

by using (1.3) with $f = \chi$ for the second equality.

Let us now consider the probability that an edge with a susceptible ego has an infectious alter. Both equations (1.8) and (1.9) depend on $\bar{p}^{\mathrm{I}}_t = \bar{N}^{\mathrm{IS}}_t / \bar{N}^{\mathrm{S}}_t$. It is thus important to obtain an equation for this quantity. In [27], this equation also leads to introduce the quantity $\bar{p}^{\mathrm{S}}_t$. From Corollary 3.5, we see that $\bar{N}^{\mathrm{S}}$ and $\bar{N}^{\mathrm{IS}}$ are differentiable and:

$$\begin{aligned}
\frac{\mathrm{d}\bar{p}^{\mathrm{I}}_t}{\mathrm{d}t} =& \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\bar{N}^{\mathrm{IS}}_t}{\bar{N}^{\mathrm{S}}_t}\Big) = \frac{1}{\bar{N}^{\mathrm{S}}_t}\frac{\mathrm{d}}{\mathrm{d}t}(\bar{N}^{\mathrm{IS}}_t) - \frac{\bar{N}^{\mathrm{IS}}_t}{(\bar{N}^{\mathrm{S}}_t)^2}\frac{\mathrm{d}}{\mathrm{d}t}(\bar{N}^{\mathrm{S}}_t) \\
=& \Big(r\bar{p}^{\mathrm{I}}_t(\bar{p}^{\mathrm{S}}_t - \bar{p}^{\mathrm{I}}_t)\theta_t \frac{g''(\theta_t)}{g'(\theta_t)} - r\bar{p}^{\mathrm{I}}_t - \beta\bar{p}^{\mathrm{I}}_t \Big) \\
&- \Big(\frac{\bar{p}^{\mathrm{I}}_t}{\theta_t g'(\theta_t)}\big(- r\bar{p}^{\mathrm{I}}_t\theta_t g'(\theta_t) + \theta_t g''(\theta_t)(-r\bar{p}^{\mathrm{I}}_t\theta_t)\big)\Big) \\
=& r\bar{p}^{\mathrm{I}}_t\bar{p}^{\mathrm{S}}_t\theta_t\frac{g''(\theta_t)}{g'(\theta_t)} - r\bar{p}^{\mathrm{I}}_t(1-\bar{p}^{\mathrm{I}}_t) - \beta\bar{p}^{\mathrm{I}}_t,
\end{aligned}$$

by using the equations 1 and 2 of (3.40) for the derivatives of $\bar{N}^{\mathrm{S}}$ and $\bar{N}^{\mathrm{IS}}$ with respect to time for the second line. This achieves the proof of (1.10).

For (1.11), we notice that $\bar{p}^{\mathrm{S}}_t = 1 - \bar{p}^{\mathrm{I}}_t - \bar{p}^{\mathrm{R}}_t$ and achieve the proof by showing that

$$\bar{p}^{\mathrm{R}}_t = \int_0^t \Big(\beta\bar{p}^{\mathrm{I}}_s - r\bar{p}^{\mathrm{I}}_s\bar{p}^{\mathrm{R}}_s\Big)\,\mathrm{d}s \tag{3.43}$$

by using arguments similar as for $\bar{p}^{\mathrm{I}}_t$. ■

Remark 2. Miller [18] shows that Volz' equations can be reduced to only three ODEs:

$$\begin{aligned}
&\bar{S}_t = g(\theta_t), \qquad \frac{\mathrm{d}\bar{R}_t}{\mathrm{d}t} = \beta\bar{I}_t, \qquad \bar{I}_t = (\bar{S}_0 + \bar{I}_0) - \bar{S}_t - \bar{R}_t, \\
&\frac{\mathrm{d}\theta_t}{\mathrm{d}t} = -r\theta_t + \beta\big(1-\theta_t\big) + \beta\frac{g'(\theta_t)}{g'(1)}.
\end{aligned}$$

The last ODE is obtained by considering the probability that an edge with an infectious ego drawn at random has not transmitted the disease. However, in his simplifications, he uses that the degree distributions $\bar{\mu}^{\mathrm{S}}_0/\bar{S}_0$ and $\sum_{k\in\mathbb{N}} p_k\delta_k$ are the same, which is not necessarily the case (see our Remark 1). Moreover, it is more natural to have an ODE on $\bar{I}_t$ and $\bar{N}^{\mathrm{IS}}_t$ is a natural quantity that is of interest in itself for the dynamics. □

A Finite measures on $\mathbb{N}$

First, some notation is needed in order to clarify the way the atoms of a given element of $\mathcal{M}_F(\mathbb{N})$ are ranked. For all $\mu \in \mathcal{M}_F(\mathbb{N})$, let F_μ be its cumulative distribution function and F_μ^{-1} be its right inverse defined as

$$\forall x \in \mathbb{R}_+,\ F_\mu^{-1}(x) = \inf\{i \in \mathbb{N},\ F_\mu(i) \geq x\}. \tag{A.1}$$

Let $\mu = \sum_{n\in\mathbb{N}} a_n\delta_n$ be an integer-valued measure of $\mathcal{M}_F(\mathbb{N})$, *i.e.* such that the a_n's are integers themselves. Then, for each atom $n \in \mathbb{N}$ of μ such that $a_n > 0$, we duplicate the atom n with multiplicity a_n, and we rank the atoms of μ by increasing values, sorting arbitrarily the atoms having the same value. Then, we denote for any $i \leq \langle \mu, \mathbf{1}\rangle$,

$$\gamma_i(\mu) = F_\mu^{-1}(i), \tag{A.2}$$

the level of the i^{th} atom of the measure, when ranked as described above. We refer to Example 1 for a simple illustration.

We now make precise a few topological properties of spaces of measures and measure-valued processes. For $T > 0$ and a Polish space (E, d_E), we denote by $\mathcal{D}([0,T], E)$ the Skorokhod space of càdlàg (right-continuous left-limited) functions from $\mathbb{R}$ to E (*e.g.* [6, 16]) equipped with the Skorokhod topology induced by the metric

$$d_T(f,g) := \inf_{\alpha \in \Delta([0,T])} \left\{ \sup_{\substack{(s,t)\in[0,T]^2,\\ s\neq t}} \left| \log \frac{\alpha(s)-\alpha(t)}{s-t} \right| + \sup_{t\leq T} d_E\big(f(t), g(\alpha(t))\big) \right\}, \tag{A.3}$$

where the infimum is taken over the set $\Delta([0,T])$ of continuous increasing functions $\alpha : [0,T] \to [0,T]$ such that $\alpha(0) = 0$ and $\alpha(T) = T$.

Limit theorems are heavily dependent on the topologies considered. We introduce here several technical lemmas on the space of measures related to these questions. For any fixed $0 \leq \varepsilon < A$, recall the definition of $\mathcal{M}_{\varepsilon,A}$ in (3.7). Remark that for any $\nu \in \mathcal{M}_{\varepsilon,A}$, and $i \in \{0, \dots, 5\}$, $\langle \nu, \chi^i \rangle \leq A$ since the support of ν is included in $\mathbb{N}$.

Lemma A.1. *Let $\mathfrak{I}$ a set and a family $(\nu_\tau, \tau \in \mathfrak{I})$ of elements of $\mathcal{M}_{\varepsilon,A}$. Then, for any real function f on $\mathbb{N}$ such that $f(k) = o(k^5)$, we have that*

$$\lim_{K\to\infty} \sup_{\tau\in\mathfrak{I}} |\langle \nu_\tau,\, f\mathbf{1}_{[K,\infty)} \rangle| = 0.$$

Proof. By Markov inequality, for any $\tau \in \mathfrak{I}$, for any K, we have

$$\sum_{k\geq K} |f(k)|\nu_\tau(k) \leq A \sup_{k\geq K} \frac{|f(k)|}{k^5},$$

hence

$$\lim_{K\to\infty} \sup_{\tau\in\mathfrak{I}} |\langle \nu_\tau,\, f \rangle| \leq A \limsup_{k\to\infty} \frac{|f(k)|}{k^5} = 0.$$

The proof is thus complete. ∎

Lemma A.2. *For any $A > 0$, the set $\mathcal{M}_{\varepsilon,A}$ is a closed subset of $\mathcal{M}_F(\mathbb{N})$ embedded with the topology of weak convergence.*

Proof. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{M}_{\varepsilon,A}$ converging to $\mu \in \mathcal{M}_F(\mathbb{N})$ for the weak topology, which implies in particular that $\lim_{n\to+\infty} \mu_n(k) = \mu(k)$ for any $k \in \mathbb{N}$. Denoting for all n and $k \in \mathbb{N}$, $f_n(k) = k^5\mu^n(k)$, we have that $\lim_{n\to+\infty} f_n(k) = f(k) := k^5\mu(k)$, μ-a.e., and Fatou's lemma implies

$$\langle \mu, \chi^5 \rangle = \sum_{k\in\mathbb{N}} f(k) \leq \lim\inf_{n\to\infty} \sum_{k\in\mathbb{N}} f_n(k) = \lim\inf_{n\to\infty} \langle \mu_n, \chi^5 \rangle.$$

Since $\langle \mu_n, \mathbf{1} \rangle$ tends to $\langle \mu, \mathbf{1} \rangle$, we have that $\langle \mu, \mathbf{1} + \chi^5 \rangle \leq A$.

Furthermore, by uniform integrability (Lemma A.1), it is also clear that

$$\varepsilon \leq \lim_{n\to\infty} \langle \mu_n,\, \chi \rangle = \langle \mu,\, \chi \rangle,$$

which shows that $\mu \in \mathcal{M}_{\varepsilon,A}$. ∎

Lemma A.3. *The traces on $\mathcal{M}_{\varepsilon, A}$ of the total variation topology and of the weak topology coincide.*

Proof. It is well known that the total variation topology is coarser than the weak topology. In the reverse direction, assume that $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of weakly converging measures belonging to $\mathcal{M}_{\varepsilon, A}$. Since,

$$d_{TV}(\mu_n,\, \mu) \le \sum_{k\in\mathbb{N}} |\mu_n(k) - \mu(k)|.$$

according to Lemma A.1, it is then easily deduced that the right-hand-side converges to 0 as n goes to infinity. ∎

Lemma A.4. *If the sequence $(\mu_n)_{n\in\mathbb{N}}$ of $\mathcal{M}_{\varepsilon, A}^{\mathbb{N}}$ converges weakly to the measure $\mu \in \mathcal{M}_{\varepsilon, A}$, then $(\langle \mu_n, f\rangle)_{n\in\mathbb{N}}$ converges to $\langle \mu, f\rangle$ for all function f such that $f(k) = o(k^5)$ for all large k.*

Proof. Triangular inequality says that:

$$\begin{aligned}|\langle \mu_n, f\rangle - \langle \mu, f\rangle| \le{}& |\langle \mu_n, f\mathbb{1}_{[0,K]}\rangle - \langle \mu, f\mathbb{1}_{[0,K]}\rangle| \\ &+ |\langle \mu, f\mathbb{1}_{(K,+\infty)}\rangle| + |\langle \mu_n, f\mathbb{1}_{(K,+\infty)}\rangle|.\end{aligned}$$

We then conclude by uniform integrability and weak convergence. ∎

Recall that $\mathcal{M}_{\varepsilon, A}$ can be embedded with the total variation distance topology, hence the topology on $\mathcal{D}([0, T], \mathcal{M}_{\varepsilon, A})$ is induced by the distance

$$\rho_T(\mu_.,\, \nu_.) = \inf_{\alpha\in\Delta([0,T])}\left(\sup_{\substack{(s,t)\in[0,T]^2,\\ s\neq t}} \left|\log\frac{\alpha(s)-\alpha(t)}{s-t}\right| + \sup_{t\le T} d_{TV}(\mu_t,\, \nu_{\alpha(t)})\right).$$

Lemma A.5. *For any $p \le 5$, the following map is continuous:*

$$\Phi_p : \begin{cases} \mathcal{D}(\mathbb{R}_+, \mathcal{M}_{\varepsilon, A}) & \longrightarrow \mathcal{D}(\mathbb{R}_+, \mathbb{R}) \\ \nu_. & \longmapsto \langle \nu_.,\, \chi^p\rangle. \end{cases}$$

Proof. It is sufficient to prove the continuity of the above mappings from $\mathcal{D}([0, T], \mathcal{M}_{\varepsilon, A})$ to $\mathcal{D}([0, T], \mathbb{R})$, for any $T \ge 0$, where the latter are equipped with the corresponding Skorokhod topologies. For μ and ν two elements of $\mathcal{M}_{\varepsilon, A}$, for any $p \le 5$, for any positive integer K, according to Markov inequality,

$$|\langle \mu,\, \chi^p\rangle - \langle \nu,\, \chi^p\rangle| \le 2\frac{A}{K^p} + |\langle \mu - \nu,\, \chi^p\mathbb{1}_{[0,\, K]}\rangle| \le 2\frac{A}{K^p} + K^p d_{TV}(\mu,\, \nu). \tag{A.4}$$

Using (A.3) and (A.4) we have for any $K > 0$:

$$d_T(\langle \mu_.,\, \chi^p\rangle,\, \langle \nu_.,\, \chi^p\rangle) \le 2\frac{A}{K^p} + K^p d_T(\mu_.,\, \nu_.),$$

and hence the continuity of Φ_p. ∎

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